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ABSTRACT (UNCLASSIFIED)

When an electromagnetic wave is scattered by an object its polarization generally changes. The change depends on the kind of object. It is therefore useful to study these changes and relate them to object features. This is the objective of polarimetry. This report serves as an introduction to polarimetry with microwaves, i.e., radar polarimetry. The subject deserves attention nowadays because of the growing number of polarimetric radars. These radars are able to measure the polarization changes caused by scattering at the earth surface.

The first chapters contains some basic electromagnetic theory relevant to the remainder of the report. The definition of polarization is the subject of the next chapter. Polarization is a two parameter quantity for which several representatons exist. Some are given, together with a few methods to generate a wave with an arbitrary polarization. The polarization of a wave can be changed by several different physical mechanisms. These are discussed in chapter 4. The scattering matrix describes the way in which the polarization of a wave is altered by scattering. Some properties and examples are given. The problem of minimization and maximization of the power received by a radar illuminating an object with a given scattering matrix is solved. The scattering matrix describes the scattering by a single stationary target; it cannot be used to describe the scattering by a time-varying or distributed target. The Stokes matrix should be used instead. The last chapter is devoted to the definition and some properties of this matrix.

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SAMENVATTING (ONGERUBRICEERD)

De polarisatie van een elektromagnetische golf verandert in het algemeen door verstrooiing aan een object. De verandering hangt af van het object. Het is daarom nuttig deze veranderingen te bestuderen en ze te relateren aan eigenschappen van het object. Dat is het uiteindelijke doel van polarimetrie. Wanneer gebruik gemaakt wordt van microgolven spreekt men van radar polarimetrie. Dit rapport is bedoeld als een introduktie in de radar polarimetrie. Dit onderwerp is actueel door het groeiende aantal polarimetrische radars. Met deze radars kunnen de door verstrooiing aan het aardoppervlak veroorzaakte polarisatie veranderingen gemeten worden.

Het eerste hoofdstuk bevat enige elementaire elektromagnetische theorie die gebruikt wordt in het vervolg van het rapport. De definitie van polarisatie wordt behandeld in het volgende hoofdstuk. Polarisatie is een twee-parameter grootheid met verschillende representaties. Enkele representaties worden gegeven, plus een paar methoden om een golf te genereren met een willekeurige polarisatie. De polarisatie van een golf kan door verschillende fysische mechanismen veranderen. Deze worden beschreven in hoofdstuk 4. De scattering matrix beschrijft hoe de polarisatie verandert door verstrooiing. Enkele voorbeelden en eigenschappen worden gegeven. Het probleem van maximalisatie en minimalisatie van het vermogen ontvangen door een radar die een object met een gegeven scattering matrix belicht wordt opgelost. De verstrooiing door een enkelvoudig stationair object wordt beschreven door de scattering matrix. Voor tijdafhankelijke of gedistribueerde doelen moet in plaats van deze matrix de Stokes matrix gebruikt worden. Het laatste hoofdstuk is gewijd aan de definitie en enkele eigenschappen van deze matrix.

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# Voorwoord

Voor remote sensing van het aardoppervlak vanuit vliegtuigen of satellieten wordt gebruik gemaakt van radarsystemen. Omdat de radar zelf voor de belichting van het oppervlak zorgt is het in principe mogelijk polarimetrisch te meten. Dat wil zeggen dat gemeten wordt hoe de polarisatie van de verstrooide golven afhangt van de polarisatie van de invallende golven. Deze afhankelijkheid verschilt per oppervlak, en bevat dus informatie over het verstrooiende oppervlak. Met een polarimetrische radar kan deze informatie worden benut. Radar polarimetrie is dat deel van de natuurkunde dat zich hiermee bezighoudt.

Dit rapport moet gezien worden als een eenvoudige inleiding in radar polarimetrie, opgebouwd vanaf de basis. Het verschaft de noodzakelijke definities van de gebruikte grootheden en geeft diverse rekenvoorbeelden. Na bestudering kan de recente literatuur over dit onderwerp begrepen worden.

# 1 THE ELECTRIC FIELD CONCEPT

In this chapter the electric field concept is introduced. Readers already familiar with this concept can skip this chapter harmlessly. Readers having a less rigorous background in physics are probably more familiar with forces than electric fields. So in order to remove the perhaps somewhat abstract character of the electric field, the relation between an electric field and the force it exerts on a charge is emphasized in this chapter.

## 1.1 The Coulomb force

The electric field is a useful concept for computations involving the Coulomb (electric) interaction. To introduce this concept the Coulomb interaction is first discussed. This leads directly to the definition of the static electric field. The discussion will only deal with static electric fields, that is, electric fields produced by charges at rest in the observer's coordinate system. Therefore the electric field does not fluctuate in time (chapter 2 deals with electromagnetic waves, i.e., time-dependent electromagnetic fields).

It is an experimental fact that two objects carrying an electric charge exert a force on each other. This force is attractive when the two charges differ in sign, and repulsive in case the charges have equal sign (both negative, or both positive). The force is caused by the Coulomb interaction, or electric interaction. Force is a vector quantity so it has a magnitude and a direction. Its magnitude and direction are given by Coulomb's law, which is expressed by a vector equation, because of the vector nature of force.

Consider the Coulomb interaction between two point-charges at rest (= static case) in the observer's coordinate system (a point-charge is a certain amount of charge confined to a very small region of space). The force one charge exerts on the other charge is given by Coulomb's law, so named after the French engineer C.A. de Coulomb (1736-1806). He was the first to state it as follows:

> The electrostatic interaction between two charged particles is proportional to their charges and the inverse of the square of the distance between them, and its direction is along the line joining the two charges.

In Fig.1.1 two (in this case both positive or negative) charges are shown, one carrying a charge $q_1$, the other carrying a charge $q_2$ (in Coulombs). The charges are seperated by a distance $|\mathbf{r}_{12}|$ (in meters). The vector $\hat{\mathbf{u}}_{12}$ is a unit vector (a vector of length 1 m), pointing from $q_1$ to $q_2$. Now the force $\mathbf{F}_{12}$ (in Newtons) that $q_1$ exerts on $q_2$ is given by the vector equation

$$\mathbf{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{\mathbf{u}}_{12}. \qquad (1.1)$$

The proportionality constant $k$ is approximately $9 * 10^9$ Nm$^2$. Note that the direction of $\mathbf{F}_{12}$ is given correctly if we use the algebraic signs for $q_1$ and $q_2$ : the force is positive (that is, repulsive) when both charges have the same sign as in Fig.1.1, and it is negative (that is, attractive) when the signs are different. The force that $q_2$ exerts on $q_1$, $\mathbf{F}_{21}$, equals $-\mathbf{F}_{12}$. So this force is of the same magnitude, but oppositely directed.

Thus far we have treated the Coulomb force between only two interacting point-charges. Suppose that a charge $q_3$ is in the presence of two other charges $q_1$ and $q_2$, as shown in Fig.1.2. Experiments show that the force on $q_3$ is just the vector sum of the separate forces on it from $q_1$ and $q_2$ . That is, the superposition principle holds for the Coulomb forces. Said differently, the force between any two charges is independent of the presence of other charges: to find the resultant force, we merely add the individual forces as vectors.

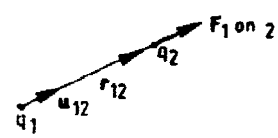


Figure 1.1: Two charges with equal sign (from [1])

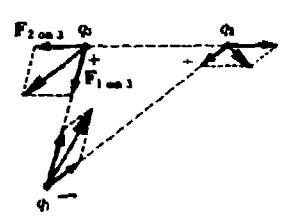


Figure 1.2: An assembly of three charges (from [1])

## 1.2 Static electric fields

Now we are ready to define the electric field. Any region where an electric charge experiences a force is thought to contain an electric field. The force is due to the presence of other charges. When these charges are at rest, they produce a static (time-independent) electric field. For example, a charge $q$ placed in a region where there are other charges $q_1, q_2, q_3, \ldots$ experiences a force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots, \tag{1.2}$$

and we say that it is in an electric field produced by the charges $q_1, q_2, q_3, \ldots$ (the charge $q$ of course also exerts forces on $q_1, q_2, q_3, \ldots$, but we are not concerned with them now). Eq.(1.2) is nothing else than a mathematical formulation of the aforementioned force superposition principle. Since the force that each charge $q_1, q_2, q_3, \ldots$ produces on the charge $q$ is proportional to $q$ (Eq.(1.1)), the resultant force $\mathbf{F}$ is proportional to $q$. Thus the force on a particle placed in an electric field is proportional to the charge of the particle. Therefore it is meaningful to define the electric field strength (or, for short, the electric field) as

$$\mathbf{E} = \frac{\mathbf{F}}{q}. \tag{1.3}$$

From this equation it is obvious that the unit in which the electric field is expressed is the N/C (or, less obvious, the V/m). Once the electric field $\mathbf{E}$ is known, we are able to compute the force on a charge $q$, simply by multiplying $\mathbf{E}$ by $q$. Note that for positive charges ($q > 0$) the direction of the force is the same as the electric field direction. For negative charges the force is oppositely directed.

The static electric field is for most charge distributions a function of the three space coordinates $x, y$ and $z$. So $\mathbf{E} = \mathbf{E}(x, y, z)$. Furthermore it is a vector field: to every point in space a vector $\mathbf{E}(x, y, z)$ is assigned. To compute the electric field for one point charge rewrite Eq.(1.1) in the form

$$\mathbf{F}_{12} = q_2\, k \frac{q_1}{r^2} \hat{\mathbf{u}}_{12}. \tag{1.4}$$

This gives the force produced by the charge $q_1$ on the charge $q_2$ placed at a distance $r$ from $q_1$. We may also say, using Eq.(1.3), that the electric field $\mathbf{E}$ at the point where $q_2$ is placed is such that $\mathbf{F} = q_2\mathbf{E}$. Therefore, by comparing both expressions of $\mathbf{F}$ we conclude that the electric field at a point 2, at a distance $r$ from a point-charge $q_1$, is given by

$$\mathbf{E}_{12} = k \frac{q_1}{r^2} \hat{\mathbf{u}}_{12}. \tag{1.5}$$

An electric field is visualized by drawing the electric field vectors for some space points. In Fig.1.3 this is done for a positive and a negative charge. Fig.1.3 is completely determined

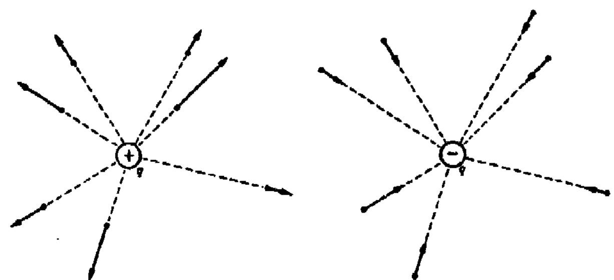


Figure 1.3: Electric field vectors of a positive (a) and a negative charge (b) (from [2])

by Eq.(1.5). The length of the electric field vector at a certain point is proportional to the inverse of the square of the distance between that point and the charge. The direction of the vectors is radially outward (inward) from the charge for a positive (negative) charge.

Another way to visualize an electric field is by drawing electric field lines. Electric field line are lines of force, which are lines that, at each point, are tangent to the direction of the electric field at that point. The direction of the field lines gives the direction of the force acting on a positive charge placed in the field. Fig.1.4 shows the same electric field as that of Fig.1.3a, but now using electric field lines. Fig.1.5 presents the electric field for

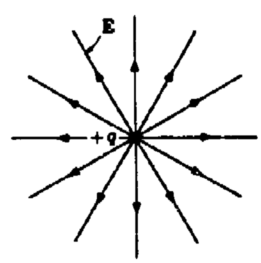


Figure 1.4: Electric field lines of a postive charge (from [1])

an assembly of two point-charges. At first sight the field line representation (Fig.1.4, 1.5) is less powerful than the vector representation (Fig.1.3), because there is no indication of the magnitude of the electric field. In Fig.1.3 the magnitude is given by the length of the electric field vectors. However, the same information is conveyed in Fig.1.4 and 1.5: it can

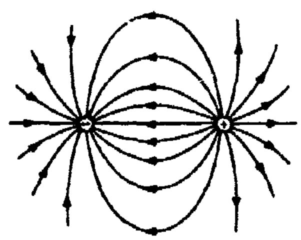

Figure 1.5: Electric field lines for an assembly of a postive and a negative charge (from [1])

be proven that the number of electric field lines passing through a small area of fixed size oriented at right angles to the electric field lines is proportional to the magnitude of the electric field.

## 1.3 Summary

It was shown in this introduction that the electric field is a vector field. This field assigns an electric field vector to every point in space. The length of an electric field vector is directly proportional to the force exerted on a point-charge by the electric field. The direction of an electric field vector equals the direction of the force exerted by the field on a positive charge. All this information is conveyed in the simple equation (1.3). An electric field is visualized by drawing electric field lines.

# 2 ELECTROMAGNETIC WAVES

A discussion of electromagnetic wave phenomena usually starts with a discussion of Maxwell's equations. These equations allow one to compute the electric and magnetic fields produced by an assembly of electric charges and currents. Because this report is not meant to be a textbook on electromagnetic theory, some of the results are simply postulated. Rigorous derivations of these results can be found in [4,8]. §2.1 is devoted to a discussion of harmonic plane waves. In §2.2 an expression for the average intensity of a plane wave is given. The energy conservation principle and the results of the first two paragraphs are used to derive a general expression for a spherical wave in §2.3.

## 2.1 Harmonic plane waves

A basic feature of Maxwell's equations for the electromagnetic field is the existence of traveling wave solutions which represent the transport of energy from one point to another. A very special, but for our purpose sufficient class of solutions is the class of harmonic plane waves (what 'harmonic plane' means is explained later on). This class is sufficient because we will deal here with electromagnetic waves produced by a harmonically excited antenna only, observed far away from the antenna or scatterer. Far away from the antenna or scatterer the wavefront of the wave is locally flat. A proof of the fact that the wavefront of a general harmonic wave is locally flat can be found in [9].

To stress the difference between the electric field of a harmonic plane wave and that of a general electromagnetic field, the equation for the general case is first given:

$$\mathbf{E}(x,y,z,t) = \hat{\mathbf{x}}E_x(x,y,z,t) + \hat{\mathbf{y}}E_y(x,y,z,t) + \hat{\mathbf{z}}E_z(x,y,z,t). \qquad (2.1)$$

The left side of the equation represents an electric vector field. It assigns an electric field vector $\mathbf{E}$ to each point in space $(x, y, z)$, possibly varying with time $t$. Because a vector can always be written as a sum of vector components multiplied by unit vectors, this is also possible for an electric field (right side of Eq.(2.1)). In this case the unit vectors are the Cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ (in this report we will always use right-handed coordinate systems). The components are the space- and time-dependent $E_x$, $E_y$ and $E_z$. Note that a general electromagnetic field of course consists of an electric and a magnetic field. However, for a discussion of polarization phenomena it is sufficient to limit the discussion to electric fields, so this will be done here.

The general form for a harmonic plane wave in vacuum, propagating in the $+\hat{\mathbf{z}}$ direction is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(\omega t - kz + \delta_x) + \hat{\mathbf{y}} E_y \cos(\omega t - kz + \delta_y) . \tag{2.2}$$

The basic features of this solution of Maxwell's equations are:

- the time- and space dependence of the electric field is conveyed in the factors $\cos(\omega t - kz + \delta_{x,y})$. So at a fixed point in space, the electric field varies as $\cos(\omega t)$. All fields exhibiting this time dependence are called *harmonic* fields. The arguments of the cosine functions are called *phases*. $\delta_x$ and $\delta_y$ are the phases of the $x-$ and $y-$component of the electric field for $z = t = 0$.
- the electric field has no $z-$component. Therefore Eq.(2.2) represents a *transverse* wave: at a fixed point in space the electric field vector is always confined to a flat plane perpendicular to the direction of propagation. $E_x$ and $E_y$ are the *positive* field amplitudes in the $x-$ and $y-$direction.
- the wave of Eq.(2.2) is a *plane* wave. A wave is called plane, when its *wavefront* is a flat plane everywhere. A wavefront is a surface in which the phase is constant (it could be called an equi-phase surface). For the wave of Eq.(2.2) the phase in a flat plane perpendicular to the direction of propagation, and at a certain time, *is* constant. This is so, because the $z-$coordinate is constant for the points in a flat plane perpendicular to the propagation direction. Hence the phases, which depend on the $z-$coordinate only at a fixed time, are the same everywhere in this flat plane. This causes the electric field to be constant also throughout the plane.
- the wave propagates in the $+z$ direction. This can be seen from the phases $\omega t - kz + \delta_{x,y}$. Suppose that the phase in a flat plane perpendicular to the propagation direction is $\phi$ for a certain $z$ and $t$. When time increases, the $z-$coordinate of the plane with phase $\phi$ increases. Therefore the wave propagates in the $+z$ direction. A wave propagating in the $-z$ direction would have phases of the form $\omega t + kz + \delta_{x,y}$.

The relation between angular frequency $\omega$ (rad/s) and frequency $f$ ($s^{-1}$) is given by $\omega = 2\pi f$. The wave number $k$ (rad/m) is related to the wavelength $\lambda$ (m) by $k = 2\pi/\lambda$. For electromagnetic waves in vacuum the relation $\lambda = c/f$ holds, with $c$ the speed of light in vacuum ($3 * 10^8$ m/s).

## 2.2 The average intensity of a harmonic plane wave

In the preceding section it was noted that traveling wave solutions represent the transport of energy from one point to another. So it should be possible to assign an intensity to a wave. The intensity is the amount of energy flowing through a unit area perpendicular to the wave's propagation direction, per unit time. It can be shown that the average intensity of a harmonic plane wave in vacuum is given by

$$I = c\epsilon_0 \mathrm{E}^2_{ave} = \frac{1}{2} c\epsilon_0 \left( E_x{}^2 + E_y{}^2 \right). \tag{2.3}$$

The average is taken, because the electric field varies within one period of the cycle, and hence the instantaneous intensity also. $\epsilon_0$ is the electric permittivity of the vacuum ($8.85 * 10^{-12}$ C/(Nm$^2$)). The dimension of intensity is J/(m$^2$s), or equivalently, W/m$^2$ (power per square meter).

## 2.3 Spherical harmonic waves

Suppose an electromagnetic wave is produced by a harmonically excited antenna. The wavefront in the vicinity of the antenna is not flat. For an antenna with vanishing dimension the wavefront is spherical. At any point in the far-field of the scatterer or antenna the radiated wave can be approximated by a plane wave whose electric field strength is the same as that of the wave and whose direction of propagation is in radial direction from the antenna. In fact, the definition of the far-field is such that this approximation is valid. As the radial distance approaches infinity, the radius of curvature of the radiated wave's wavefront also approaches infinity and thus in any specified direction the wave appears locally as a plane wave. For antennas whose maximum overall dimension $D$ is large compared to the wavelength $\lambda$, the far-field region is commonly taken to exist at distances greater than $2D^2/\lambda$.

So the equation for a spherical wave in the far-field region is also given by Eq.(2.2), possibly multiplied by a factor. In this case $x$ and $y$ refer to the coordinates in a local coordinate system at the observation point. The $x-$ and $y-$axis are contained in a plane normal to the propagation direction (Fig.2.1). Our aim is to find the multiplicative factor just mentioned. To do this, we recall that for a plane wave the energy flow per second through an area $A$ perpendicular to the direction of propagation at a distance $r$ is given by (Eq.(2.3), Fig.2.1):

$$AI = \frac{A}{2} c\epsilon_0 \left( E_x{}^2 + E_y{}^2 \right). \tag{2.4}$$

Because the range $r$ and area $A$ are related by $A = \Omega r^2$ , where $\Omega$ is the solid angle subtended by $A$, this can be rewritten as

$$AI = \frac{\Omega r^2}{2} c\epsilon_0 \left( E_x{}^2 + E_y{}^2 \right). \tag{2.5}$$

Let the area $A'$ be defined by the requirement that it subtends the same solid angle as $A$, and is located at a distance $r' > r$ (shifted along the direction of propagation). So $A' = \Omega r'^2$. The conservation of energy principle requires that the same energy flows through $A'$ and $A$, because they subtend the same solid angle. Therefore it can be concluded that the electric

field components $E_x$ and $E_y$ have to fall off as $1/r$. Hence the electric field of a spherical wave is given by (compare Eq.(2.2))

$$\mathbf{E}(z,t) = \frac{1}{kr}\ [\hat{\mathbf{x}}E_x \cos(\omega t - kz + \delta_x) + \hat{\mathbf{y}}E_y \cos(\omega t - kz + \delta_y)] \qquad (2.6)$$

locally, with $z$ in the radial direction. The wavenumber $k$ is added in the denominator to ensure that the equation is dimensionally correct.

The foregoing discussion is not only valid for electromagnetic waves produced by antennas, but for all electromagnetic waves produced by a source of finite size. Far enough away from the source (which can be an object scattering an electromagnetic wave) the wave will appear to be flat locally, while its field amplitude falls of as $1/r$.

## 2.4 Summary

In this chapter we first discussed harmonic plane waves. This special class of travelling waves is very important, because all waves are locally plane when observed at great distances from (or equivalent: in the far-field of) the source that produced them. Secondly, an expression was given for the average intensity of a plane wave. This expression, together with the conservation of energy principle, was used to show that the electric field amplitude of a spherical wave has to decrease as $1/r$ in the far-field.

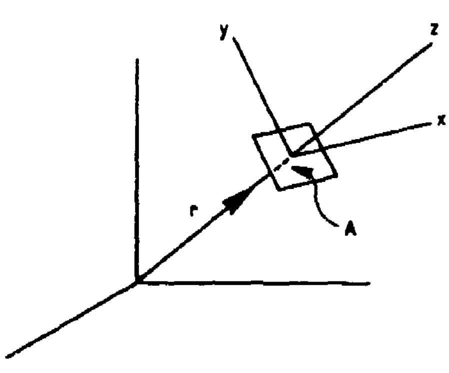


Figure 2.1: Local coordinate system

# 3 THE POLARIZATION OF AN ELECTROMAGNETIC WAVE

In the preceding chapter it was shown that electromagnetic waves are transverse waves. This means that the electric field (vector) is always perpendicular to the direction of propagation. As for all transverse waves, this leads to the introduction of the polarization concept. In §3.1 it is shown that the polarization of a wave is determined by two parameters. The polarization can also be represented by a complex vector with two components (§3.2). §3.3 is devoted to the polarization of antennas. To remove the somewhat abstract character of the polarization concept, §3.4 is devoted to a description of two simple methods to generate an arbitrarily polarized wave.

## 3.1 The polarization ellipse

In chapter 2 the equation of the electric field of a plane electromagnetic wave propagating along the positive z-axis was given as (2.2)

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(\omega t - kz + \delta_x) + \hat{\mathbf{y}} E_y \cos(\omega t - kz + \delta_y)\,. \tag{3.1}$$

To gain a better understanding of the behaviour of the electric field it is helpful to graph the variation of the electric field vector as a function of time. Because an electromagnetic wave is a transverse wave, the electric field vector is always contained in a plane perpendicular to the direction of propagation. Therefore the variation of the electric field as a function of time is most conveniently graphed in a plane perpendicular to the direction of propagation. This is done in Fig. 3.1 for one period of oscillation (of duration $2\pi/\omega$). In one oscillation period the trajectory traversed by the tip of the electric field vector is generally an ellipse, called the polarization ellipse. The handedness (direction of traversal) is indicated by the arrow, in this case clockwise or right-handed. The positive amplitude of the electric field in the $x-$ ($y-$) direction is $E_x$ ($E_y$). It is important to realize that the same ellipse is traced out in every plane perpendicular to the propagation direction. The shape of the ellipse and the handedness (thus not including its dimension) determine the polarization which is constant throughout the whole of space for a plane wave. The polarization is characterized by two parameters:

1. the orientation $\psi$. This is the orientation of the longer axis of the ellipse with respect to the positive $x$-axis (range: 0 to 180°).
2. the ellipticity $\chi$ (range: -45 to 45°). The ellipticity is a measure for the 'fatness' of the ellipse. When $\chi$ equals zero, the ellipse is degenerated to a line. When $\chi$ equals

±45°, the electric field traces out a circle. The handedness of the ellipse is given by the sign of the ellipticity. Positive ellipticities correspond to left-handedness, negative ellipticities to right-handedness.

These parameters (within the specified ranges) are sufficient to generate any possible ellipse, including the handedness. Note that the handedness depends on the look-direction, i.e., when the ellipse is observed from the other side (looking towards the approaching wave) the handedness is opposite (in our case, left- instead of right-handed). The most often used convention is to look into the direction of propagation (see for example the IEEE standard [10]). This convention will also be used throughout this report.

Thus the polarization of a plane wave is in general characterized by two parameters, the orientation $\psi$ and ellipticity $\chi$. The dimension of the ellipse is not needed, because it is related to the intensity of the wave. When the polarization ellipse is degenerated to a straight line (circle) the wave is called linearly (circularly) polarized. Linear polarizations result when $\chi = 0°$. Special cases of linear polarizations are horizontal ($\psi = 0°$, electric field aligned along the the $x$-axis) and vertical ($\psi = 90°$, electric field aligned along the $y$-axis) linear polarization. The condition for circular polarization is that $\chi$ equals ±45° (see table 3.1).

Up to now we have only studied the curve traced out *in time* by the electric field vector in a fixed plane perpendicular to the direction of propagation. Fig.3.2 shows the curve in three-dimensional space for a right-handed circularly polarized wave, *at a fixed instant in time*. In general such a curve is an elliptical helix. Note that right- (left-) handed waves

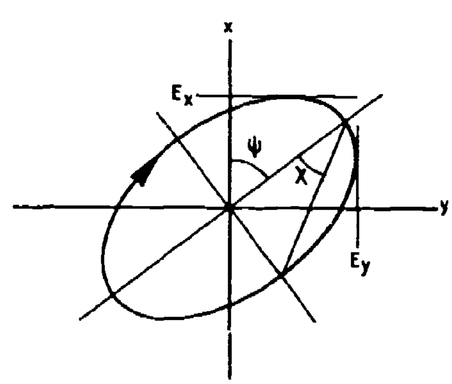


Figure 3.1: Polarization ellipse

| polarization | orientation $\psi$ | ellipticity $\chi$ | pol. vector $\hat{p}$ | pol. factor $\rho$ |
|---|---|---|---|---|
| linear horizontal | 0° | 0 | $\begin{pmatrix}1\\0\end{pmatrix}$ | 0 |
| linear vertical | 90° | 0 | $\begin{pmatrix}0\\1\end{pmatrix}$ | $\infty$ |
| right-hand circular | $\star$ | −45° | $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ | $-i$ |
| left-hand circular | $\star$ | +45° | $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ | $i$ |

Table 3.1: Some common polarizations ($\star$ means 'not determined')

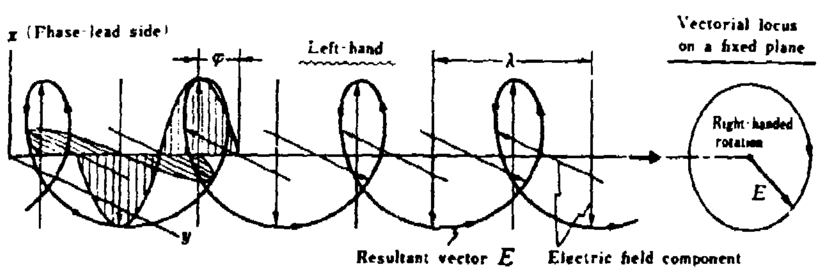


Figure 3.2: Circular helix

trace out a left- (right-) handed helix. At first sight this seems to be in contradiction with the IEEE definition [10]. But one should remember that the IEEE definition used in this report is based on the handedness of the polarization ellipse in a *fixed* plane perpendicular to the propagation direction, looking in that direction. So to determine the handedness of the wave of Fig.3.2, we have to 'push' the helix in the propagation direction through a fixed plane perpendicular to the $z$-axis, and look into the propagation direction. Then it becomes clear that the point of intersection of helix and plane traverses a right-handed circle.

All possible polarizations can be displayed through a mapping of each polarization ellipse onto the *Poincaré sphere* (Fig.3.3). An elliptical polarization with orientation $\psi$ and ellipticity $\chi$ is mapped onto a point with longitude $2\psi$ and latitude $2\chi$ on the sphere. The equator of the sphere thus contains all linear polarizations, the poles circular polarizations etc. All left-handed (right-handed) polarizations map onto the northern (southern) hemisphere. The distance of a point to the origin of the sphere can be used as a measure for

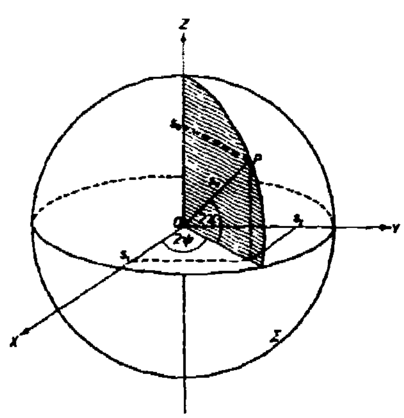


Figure 3.3: Poincaré sphere (from [3])

the intensity of the wave. Waves with the same intensity are then mapped onto the same sphere. When the polarization of waves and their intensities are displayed in this way it could be said that the mapping takes place in Poincaré space.

The next problem to address is how the orientation and ellipticity of a wave are related to the parameters of Eq.(3.1). On investigating Eq.(3.1) we can draw the next two conclusions:

1. the relation depends on the phase difference $\delta_y - \delta_x$ only, not on the individual absolute phases. This has to be the case, because adding a constant to both absolute phases

doesn't change the shape of the ellipse - it only influences the exact location of the tip of the electric field vector on the ellipse at a certain moment.

2. the relation depends on the quotient $E_y/E_x$ only, not on the individual field components. This is so, because on multiplying Eq.(3.1) by a constant the intensity of the wave changes only, not the shape of the polarization ellipse.

The deriviation of the relation between the parameters of Eq.(3.1) and the geometrical parameters $\psi$ and $\chi$ is lengthy rather than difficult, so only the results are given here. The deriviation can be found in [3]. It leads to

$$\tan(2\psi) = \tan(2\alpha)\cos\delta \quad = \quad \frac{2E_xE_y}{E_x^2 - E_y^2}\cos\delta \tag{3.2}$$

$$\sin(2\chi) = \sin(2\alpha)\sin\delta \quad = \quad \frac{2E_xE_y}{E_x^2 + E_y^2}\sin\delta \tag{3.3}$$

where

$$\tan\alpha = \frac{E_y}{E_x} \qquad (0 \leq \alpha \leq \frac{\pi}{2}) \tag{3.4}$$

$$\delta = \delta_y - \delta_x \qquad (0 \leq \delta < 2\pi). \tag{3.5}$$

From these equations it is clear that the orientation and ellipticity depend on $\delta_y - \delta_x$ and $E_y/E_x$ only, as predicted. Note that $|\tan\chi|$ equals the length of the minor axis divided by the length of the major axis of the polarization ellipse.

## 3.2 The polarization vector

In computations it is often easier to characterize the polarization of a wave by a *polarization vector*, than by the orientation and ellipticity. The polarization vector for a wave is found as follows. First we rewrite Eq.(3.1) in vector form as

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{ \begin{pmatrix} E_x e^{i\delta_x} \\ E_y e^{i\delta_y} \end{pmatrix} e^{i(\omega t - kz)} \right\}. \tag{3.6}$$

'Re' means 'take the real part of'. As is customary with time harmonic problems, we drop the exponential propagation factor and the 'Re' operator. This leads to the complex two-dimensional polarization vector

$$\mathbf{p} = \begin{pmatrix} E_x e^{i\delta_x} \\ E_y e^{i\delta_y} \end{pmatrix}. \tag{3.7}$$

The polarization vector depends on the four parameters $E_x, E_y, \delta_x$ and $\delta_y$. Because the polarization is determined by only two parameters the number of parameters of Eq.(3.7) should be reduced to two. As noted in the preceding paragraph, the polarization depends on $\delta_y - \delta_x$ and $E_x/E_y$ only. Therefore the polarization represented by Eq.(3.7) is not changed when the polarization vector is multiplied by a (possibly complex) factor. This factor is (not uniquely) determined by demanding that $\mathbf{p} \cdot \mathbf{p}^* = 1$. The $\cdot$ denotes the standard vector inner product, $^*$ complex conjugation. When this holds for a polarization vector, the vector is said to be *normalized*. To indicate that a polarization vector is normalized it is customary to put a $\hat{}$ on top of it, as in $\hat{\mathbf{p}}$. A normalized polarization vector can no longer be used to calculate the wave intensity (see Eq.(2.3)). A normalized vector corresponding to Eq.(3.7) is

$$\hat{\mathbf{p}} = \frac{1}{\sqrt{E_x^2 + E_y^2}} \begin{pmatrix} E_x \\ E_y e^{i(\delta_y - \delta_x)} \end{pmatrix}. \tag{3.8}$$

When this vector is multiplied by a complex constant with magnitude 1 it is still normalized, according to the definition.

Since the polarization depends on two parameters it is possible to define the polarization by only one complex number. This complex number is called the *polarization factor* $\rho$, and is defined as

$$\rho = \frac{E_y}{E_x} e^{i(\delta_y - \delta_x)}.$$

When polarization factors are used derivations sometimes become surprisingly simple.

Table 3.1 presents some common polarizations along with the corresponding orientation and ellipticity, the normalized polarization vector and the polarization factor.

## 3.3 Antenna polarization

In the far-field region of an antenna the electric field is given by Eq.(2.6). Because the electric field of a plane and a spherical wave differ by the factor $1/(kr)$ only, the theory and definitions presented in the preceding two paragraphs are also valid for a spherical wave in the far-field region. The polarization of an antenna is now defined as the polarization of the far field transmitted by this antenna, *regardless whether it is used for transmitting or receiving*. In general this polarization depends on the direction of the observation point

with respect to the antenna. By convention, when the direction is not stated, the antenna polarization is taken to be the polarization in the direction of maximum gain.

For example, for a vertically oriented dipole (Fig.3.4) the direction of maximum gain is in the $y$-axis direction. The normalized polarization vector is

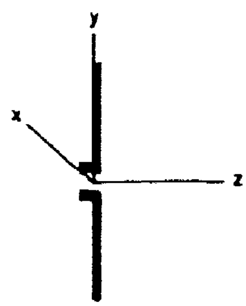


Figure 3.4: Dipole antenna

$$\hat{\mathbf{p}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.9}$$

So the electric field is vertically polarized. This complies with what is expected for an antenna in which current can only flow vertically.

It is obvious that a horizontally polarized wave incident on this vertical dipole will not produce a voltage at the output terminals of the dipole. It is also intuitively clear that this voltage is a maximum when a vertically polarized wave is incident. So the voltage depends on the polarization of the incident wave. An interesting problem is to find the wave polarization needed to generate maximum power at the terminals of an arbitrarily polarized antenna (commonly called the maximum polarization). To solve this problem we proceed as follows: assume we have an arbitrarily polarized antenna at our disposal. On transmitting an electromagnetic wave is propagated from the antenna towards a point A. The wave's polarization is characterized by an orientation $\psi$ and ellipticity $\chi$ (Fig.3.1). The ellipse is drawn as if it is observed from the antenna postion. Suppose now that we reverse time. The wave first emerging from the antenna is now incident on the antenna. If we want to determine the polarization of this wave, we have to observe it from point A, looking towards the antenna. The polarization ellipse observed is given in (Fig.3.5). Its polarization is given by

$$\psi_{\text{matched}} = \pi - \psi_{\text{antenna}} \tag{3.10}$$

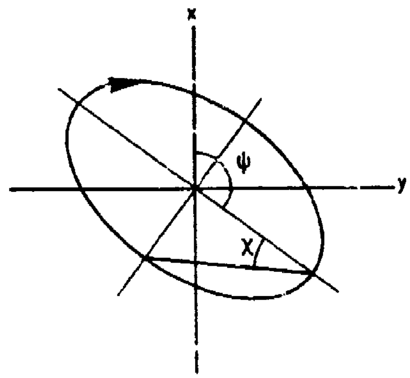


Figure 3.5: Polarization ellipse as observed from point A

$$\lambda_{\text{matched}} = \lambda_{\text{antenna}}. \tag{3.11}$$

The handedness has not changed because it reversed twice: once because of the change of observation point (from antenna to point A), and once because of the time reversal.

The only thing left to proof is that the polarization of Fig.3.5 is the maximum polarization we were looking for. The first thing to recall is that the reciprocity principle [11] states that the power needed to generate the transmitted wave equals the power generated in the antenna load on receive of this very same wave. Secondly, it is clear this is also the maximum power that can ever be received, because it is simply not possible to receive more power than is put in the wave on transmit. So Eqs.(3.10) and (3.11) indeed represent the maximum polarization. The wave and antenna are said to be 'polarization-matched' in this case.

We can now also compute the polarization vector corresponding to Eqs.(3.10) and (3.11). Substituting these equations in Eqs.(3.2) and (3.3) reveals that $\alpha$ remains the same, but $\delta_{\text{matched}}$ becomes $\pi - \delta$. After substitution of these results in Eq.(3.8), it follows that an antenna with polarization vector $\hat{\mathbf{p}}$ delivers maximum power into its load when a wave with polarization $\hat{\mathbf{p}}_m$ is incident on it, where

$$\hat{\mathbf{p}}_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{p}}^*. \tag{3.12}$$

Another polarization to be used in the remainder of this report is the *orthogonal polarization.* Two antennas with polarizations given by $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}_\perp$ are orthogonally polarized when the relation

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{a}}_\perp^* = 0 \tag{3.13}$$

holds. If the polarization $\hat{\mathbf{a}}$ has orientation $\psi$ and ellipticity $\chi$, the orientation $\psi_\perp$ and ellipticity $\chi_\perp$ of $\hat{\mathbf{a}}_\perp$ are given by

$$\begin{aligned} \psi_\perp &= \psi \pm \frac{\pi}{2} \\ \chi_\perp &= -\chi. \end{aligned}$$

where $\psi_\perp$ has to be between 0 and $\pi$. When a polarization and its associated orthogonal polarization are mapped onto the Poincaré sphere, these points are antipodal on the sphere.

## 3.4 Generation of an arbitrarily polarized wave

An arbitrarily polarized wave can be generated in several ways. In this paragraph we will investigate two methods. The first uses only one moving point charge. This method is not very practical, but provides some insight needed to understand the remainder of this report. For the second, more practical method, two identical orthogonal dipole antennas are needed.

### 3.4.1 The field produced by a harmonically oscillating point charge

Let the position of a moving point charge as a function of time be given by

$$\mathbf{\Psi}(t) = \hat{\mathbf{x}} x_0 \cos(\omega t + \delta_x) + \hat{\mathbf{y}} y_0 \cos(\omega t + \delta_y).$$

So the charge oscillates harmonically in two dimensions in the $xy$-plane. It follows an elleptical trajectory in general. This motion will give rise to an electromagnetic wave. In [4] it is shown that at sufficient great distances from the charge (in the far-field region), its time-dependent electric field is given by

$$\mathbf{E}(z,t) = -\frac{q\ddot{\mathbf{\Psi}}(t - \frac{z}{c})}{zc^2}$$

for a point on the $z$-axis with coordinates $(0,0,z)$. $q$ is the charge of the point-charge, $c$ the velocity of light. Thus, since $\ddot{\Psi} = -\omega^2\Psi$, we have

$$\mathbf{E}(z,t) = \frac{q\omega^2\Psi(t-\frac{z}{c})}{zc^2}.$$

Note that this result is in accordance with the deriviation of §2.3: the electric field amplitude of this wave decreases as $1/r$ in the far-field region. Aside from a proportionality constant, the tip of the electric field vector mimics the movement of the particle. So, when the point-charge oscillates along the $x$-axis, the electric field is linearly polarized along the $x$-axis. In general, if the point-charge traverses an ellipse characterized by a certain orientation and ellipticity, then the far-field produced by the charge is polarized with the same orientation and ellipticity. Even if an electric field is not really produced by a single charge $q$, one can think of the field as being produced by a charge $q$ moving in the proper way (lacking any explicit knowledge of the source of the field, one cannot tell that it is *not* produced by the 'effective' point charge $q$).

It is also allowed to reverse the conclusion: when a wave is incident on a point-charge, the oscillation of the charge mimics the movement of the tip of the electric field vector of the incident wave (when we neglect the fact that the charge radiates because of this oscillation). In [4] it is shown that this also holds for a bound charge, when damping effects are neglected.

### 3.4.2 The field produced by two orthogonal dipoles

Another way to generate an arbitrarily polarized field utilizes two orthogonal dipoles. These two dipoles are drawn in Fig.3.6. One is oriented horizontally (along the $x$-axis), the other vertically (along the $y$-axis). The distance between the dipoles' centres is $l$. It is assumed the dipoles are fed by a source of angular frequency $\omega$. Furthermore, the lengthes of the feeding cables are adapted in such a way that the phases of the signals are equal upon arrival at the dipoles. We are free to choose these phases to be zero. The signals do not necessarily have equal amplitudes.

The centres of the two dipoles and the point $A$ (which is in the far-field of both dipoles) are situated on a straight line. The distance of $A$ to the nearest dipole (aligned with the $x$-axis) is $z$. When we assume that the field of one dipole is not disturbed by the other dipole, the electric field produced by the two dipoles at point $A$ is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}E_x\cos(\omega t - kz) + \hat{\mathbf{y}}E_y\cos(\omega t - k(z+l)). \tag{3.14}$$

So the normalized complex polarization vector is

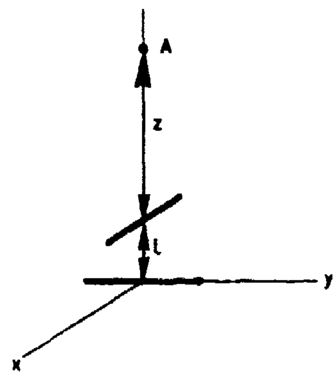


Figure 3.6: Two dipole antennas

$$\hat{\mathbf{p}} = \frac{1}{\sqrt{E_x^2 + E_y^2}} \begin{pmatrix} E_x \\ E_y e^{-ikl} \end{pmatrix}. \tag{3.15}$$

It is now obvious that any polarization can be produced by such an assembly of two dipoles. For example, right-handed circular polarization is generated by choosing $E_x = E_y$ and $l$, the distance between the dipoles, $\lambda/4$. The polarization vector is now

$$\hat{\mathbf{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{3.16}$$

## 3.5 Summary

In this chapter the polarization concept was introduced. It was shown that the polarization depends on two parameters, the orientation and ellipticity of the ellipse, traced out by the electric field vector. For computational efficiency it is desired to use (normalized) polarization vectors. These are vectors determined by two independent parameters after normalization. The polarization of an antenna was defined as the polarization of the far field produced by this antenna on transmit. Starting from general considerations we were able to determine the polarization of a wave that is received 'best' by an antenna. In the last paragraph two methods to generate an arbitrarily polarized wave were given.

# 4 CHANGE OF POLARIZATION

When an electromagnetic wave interacts with matter, it most often happens that its polarization is changed by the interaction. Because polarization is a two parameter quantity, the polarization change can be accomplished in essentially two different ways, or a combination of both. The first way is to change the field amplitude ratio $E_y/E_x$, the second alteration of the phase difference $\delta_y - \delta_x$. § 4.1 presents some simple examples of the two different cases. § 4.2 illustrates the combination of both cases, by a discussion of scattering at a plane interface between two media.

## 4.1 Simple examples

### 4.1.1 Alteration of the field amplitude ratio

Consider Fig.4.1. It shows a grid of parallel conducting wires stretched along the $y$-direction.

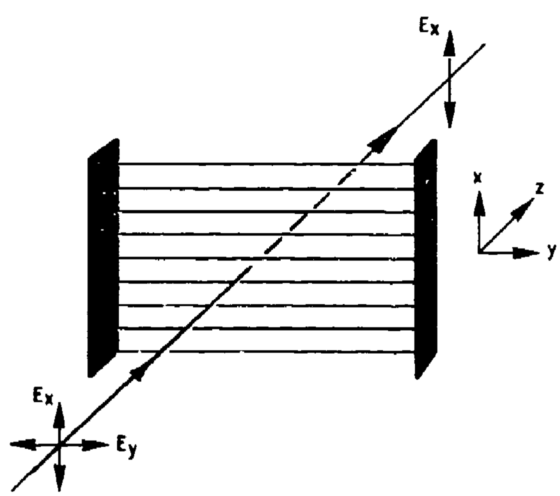


Figure 4.1: Wire grid absorbs $y$-component of microwaves (from [4])

Suppose that an electromagnetic wave (with wavelength in the order of the wire distance) is incident with non-vanishing $x$- and $y$-components $E_x$ and $E_y$. We may consider the effect of the wires on the components separately. First consider the $y$-component, along the wires. The free electrons are driven along the wire by the electric field. The electric field does work on the electrons; they transfer some of their energy to the copper lattice through collisions. The electrons also radiate. It turns out that their radiation in the forward direction interferes destructively with the incident radiation and cancels it to zero. In the backward direction, the radiation due to motion of the electrons along $\hat{\mathbf{y}}$ gives a reflected wave (the wire grid behaves like a mirror for the radiation). Thus the grid eliminates the $y$-component $E_y$.

Now consider what happens along $\hat{\mathbf{x}}$. For thin wires the electrons are not free to move in this direction. So they do not absorb energy, nor do they radiate. Consequently the $x$-component of the incident wave is unaffected. It can be concluded that the wave emanating from the wire grid is vertically polarized, independent of the polarization of the incident radiation.

The well-known polaroid behaves somewhat like a wire grid. In this case a wire is formed by a long hydrocarbon chain carrying conduction electrons. The electrons are free to move along the wire, but not perpendicular to it.

### 4.1.2 Alteration of the phase difference

The phase difference between the $x$- and $y$-component of an arbitrarily polarized wave can simply be altered by traversal of certain anisotropic media. An anisotropic medium is a medium in which a physical quantity depends on the direction in which it is measured. Consider a material for which the index of refraction $n$ for electric fields in the $x$-direction exceeds that for fields in the $y$-direction. Because the velocity $v$ of propagation of an electromagnetic wave is related to $n$ by $v = c/n$, the velocity $v_x$ of the $x$-component of an electromagnetic wave propagating along the $z$-axis is smaller than the velocity $v_y$ of the $y$-component. Eq.(2.2) should thus be rewritten as

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(\omega t - k_x z + \delta_x) + \hat{\mathbf{y}} E_y \cos(\omega t - k_y z + \delta_y)\,, \tag{4.1}$$

where $k_x$ exceeds $k_y$. The polarization vector corresponding to this wave is

$$\mathbf{p} = \begin{pmatrix} E_x e^{i\delta_x} \\ E_y e^{i((k_x - k_y)z + \delta_y)} \end{pmatrix} . \tag{4.2}$$

The polarization is thus a function of $z$.

Cellophane is a material for which the wave velocities in orthogonal directions differ generally. An interesting difference exists between polaroid and cellophane: polaroid decreases

the intensity of a wave passing through it (it appears to be dark), while cellophane does not (it is transparent). This is because polaroid changes the field amplitude, while cellophane does not. And the intensity of a wave depends on the field amplitude, not on the phase difference between orthogonal components.

Another case where the intensity of the wave is not changed, but only its polarization, is called *Faraday rotation* [12]. This phenomenon was discovered by Faraday back in 1845. He discovered that when a magnetic field was applied to a transparent substance (like glass) the orientation of the polarization was rotated through an angle depending on the strength of the magnetic field and the distance traversed in the substance. This is also what happens in the ionosphere. The ionosphere is that region of the earth's atmosphere lying approximately between 50 km and one earth radius (6370 km). On traversing the ionosphere the orientation of the polarization of a microwave is changed. The amount of rotation is proportional to $1/f^2$ for propagation parallel to the magnetic field, where $f$ denotes the frequency of the microwaves. For microwaves with frequencies lower than about 3 GHz the amount of rotation is not negligible. This has to be taken into account when a low frequency microwave signal should be received properly. Possible solutions are to use circularly polarized antennas (Faraday rotation obviously does not affect circularly polarized waves), or to use two orthogonally linearly polarized receive antennas.

## 4.2 Scattering at a plane interface

When a plane wave is incident from above at a plane interface between two media, part of the wave will penetrate refracted into the second medium, and part of it will be reflected. A sketch of this situation is given in Fig.4.2. Let us assume that the upper medium is air, having a permeability $\mu_0$ and permittivity almost $\epsilon_0$. The underlying medium is assumed to be non-magnetic, so its permeability is also $\mu_0$. Its complex permittivity is given by

$$\epsilon \equiv \epsilon_0 \epsilon_r \equiv \epsilon_0 \left( \epsilon_r' - i\epsilon_r'' \right) = \epsilon_0 \left( \epsilon_r' - i\frac{\sigma}{\epsilon_0 \omega} \right).$$

$\epsilon_0$ is the permittivity of the vacuum and $\epsilon_r$ the relative complex permittivity with real part $\epsilon_r'$ (the dielectric constant) and complex part $\epsilon_r''$. $\sigma$ denotes the conductivity and $\omega$ the angular frequency.

By applying Maxwell's equations to the problem it is possible to derive Snell's well-known laws of reflection and refraction [3]. In this report we are interested in the polarization properties of the reflected wave. To analyze this we split the incident and reflected waves in two linearly polarized components each. The first component is parallel to the plane of incidence, where the plane of incidence is defined as the plane containing the direction of the incident wave and the normal to the interface. In the figure the plane of incidence equals the plane of the paper. The electric field of the parallel component is labelled $E_{1\|}$. This is a complex number, consisting of the electric field's amplitude and phase. The second

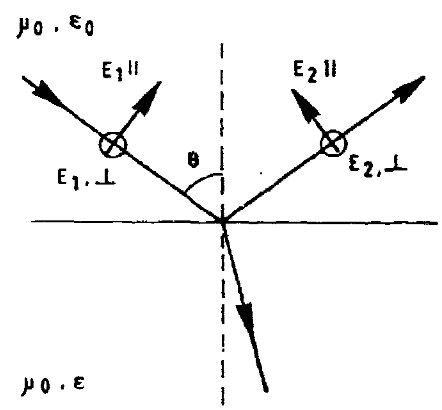


Figure 4.2: Reflection at a plane interface between two media

component is perpendicular to the plane of incidence. It is labelled $E_{1\perp}$. In Fig.4.2 the parallel component is indicated by an arrow in the plane of the paper. Its direction is important. It is simplest to think of the direction as indicating the direction of field vector during the first half period of a wave train (when the direction is reversed, this will indicate a phase shift $\pi$). The perpendicular component is indicated by a $\oplus$ or $\odot$, depending on whether the field vector is pointing into the paper. or out of the paper (during the first half period of a wave train).

The electric fields $E_{2\parallel}$ and $E_{2\perp}$ of the reflected wave are related to the electric fields of the incident wave by

$$E_{2\parallel} = R_{\parallel} E_{1\parallel} \tag{4.3}$$
$$E_{2\perp} = R_{\perp} E_{1\perp}, \tag{4.4}$$

where $R_{\parallel}$ and $R_{\perp}$ are complex reflection coefficients (also known as Fresnel reflection coefficients). It can be shown that the respective reflection coefficients $R_{\parallel}$ and $R_{\perp}$ are given by

$$R_{\parallel} = \frac{\epsilon_r \cos\theta - \sqrt{\epsilon_r - \sin^2\theta}}{\epsilon_r \cos\theta + \sqrt{\epsilon_r - \sin^2\theta}} \tag{4.5}$$

$$R_{\perp} = \frac{\cos\theta - \sqrt{\epsilon_r - \sin^2\theta}}{\cos\theta + \sqrt{\epsilon_r - \sin^2\theta}}. \tag{4.6}$$

In this equation $\theta$ is the incidence angle (Fig.4.2). The reflection coefficients are generally complex, because $\epsilon_r$ is complex. From Eqs.(4.3) and (4.4) it follows that the absolute value of $R$ equals the ratio of the amplitudes of reflected to incident wave, whilst the argument equals the phase shift caused by the reflection at the interface. Because the Fresnel coefficients are generally complex, both the amplitude ratio and the phase difference of an incident wave are affected.

Fig.4.3 shows the polarization of the reflected wave when a circular wave is incident at the earth surface for several incidence angles.

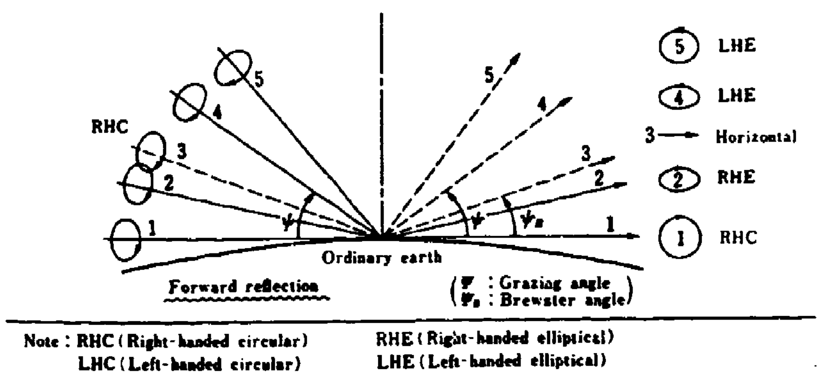


Figure 4.3: Reflection by the earth surface

An interesting situation arises if the incidence angle equals $\theta_B = \tan^{-1}\left(\sqrt{\epsilon_r}\right)$. It can be easily seen from Eq.(4.5) that $R_{\parallel}$ vanishes for this angle $\theta_B$, when medium B is non-conducting (i.e., its relative permittivity $\epsilon_r$ is real). Therefore the only component remaining is the perpendicular component, resulting in linear polarization parallel to the earth's surface (horizontal polarization). $\theta_B$ is called the Brewster angle after the discoverer of its significance. If the medium is conducting, indicated by a complex permittivity $\epsilon_r$, $R_{\parallel}$ cannot be made to vanish, as is also easily checked from Eq.(4.5). However $R_{\parallel}$ will attain a minimum for some incidence angle, which is sometimes referred to as the quasi-Brewster angle.

For a metallic medium the conductivity $\sigma$ approaches infinity. From the equations for the reflection coefficients it is easily shown that in this case $R_{\parallel} = +1$, while $R_{\perp} = -1$. So the perpendicular component of the incident wave is subject to a phase shift $\pi$. This result will be used later on in this report to derive the scattering matrix for a metallic dihedral (a device consisting of two metal plates connected at right angles to each other).

## 4.3 Summary

This chapter focussed on mechanisms able to change the polarization of a wave. The first two simple examples involved a wire grid and anisotropic media respectively. The wire grid absorbed/reflected one of the components of the electric field while the phase difference between the components remained unchanged. On the other hand, the anisotropic media considered altered the phase difference only. The polarization change caused by scattering at an interface between air and a non-magnetic medium was given by the Fresnel reflection coefficients. Because these coefficients are complex in general, both the phase difference and amplitude of the incident wave are changed.

# 5 THE SCATTERING MATRIX

The previous chapter was devoted to a discussion of mechanisms able to change the polarization of an electromagnetic wave. One of the mechanisms was scattering. The scattering matrix to be introduced in this chapter is a mathematical quantity describing the polarization change caused by scattering by an object. Besides a description of some general properties of the scattering matrix, specific examples are given for a flat plate, a dihedral and a trihedral. The derivation of the scattering matrix of a dihedral utilizes the theory presented in the preceding chapter.

## 5.1 Transmission between arbitrarily polarized antennas

Consider a transmit antenna A with polarization given by $\hat{\mathbf{a}}$ and a receive antenna B with polarization given by $\hat{\mathbf{b}}$. The voltage at the terminals of B caused by the field transmitted by A depends on parameters independent of the antenna polarizations, like antenna gains, wavelength and distance, as well as on the polarizations $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. In fact, the voltage is given by

$$V = c\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}_m^* = c\hat{\mathbf{a}} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{b}}. \tag{5.1}$$

The constant $c$ accounts for all the polarization-independent parameters, like the distance between the antennas.

Although Eq.(5.1) is simply postulated here, a few observations can be made to make its validity plausible. For example, in chapter 2 it was shown that a wave with polarization $\hat{\mathbf{b}}_m$ is polarization matched to an antenna with polarization $\hat{\mathbf{b}}$. So substition of $\hat{\mathbf{b}}_m$ for $\hat{\mathbf{a}}$ in Eq.(5.1) should maximize $V$. This is indeed the case because it turns out that $V = c$. $V$ can never exceed $c$, because the inner product of two normalized polarization vectors is 1 at most. A second observation to be made is that Eq.(5.1) satisfies the reciprocity principle [11]. For antennas this principle states that the voltage $V$ is the same whether antenna A is used for transmitting and B for receiving, or vice versa. So the relation

$$c\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}_m^* = c\hat{\mathbf{b}} \cdot \hat{\mathbf{a}}_m^*$$

should hold. The proof of this equality is left as an exercise to the reader.

At first sight the voltage $V$ should be maximum when identical antennas A and B are used. This is not generally so. Consider two dipoles oriented at 45° with respect to the horizon

$(\psi = 45^{\circ}, \chi = 0^{\circ})$. So

$$\hat{\mathbf{a}} = \hat{\mathbf{b}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$V$ is then given by

$$V = c\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{c}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{c}{2}(1 \cdot 1 + 1 \cdot -1) = 0.$$

This is a consequence of our polarization definition: whether an antenna is used for receiving or transmitting, the antenna polarization is always defined as the polarization of the wave transmitted by the antenna. The above derived result is in accordance with the observation that the receive antenna is oriented normally to the transmitting antenna.

Finally, it is easily shown that $V = c$ when two identical circularly polarized antennas are used. So in this case the use of two identical antennas results in maximum voltage at the terminals of antenna B.

## 5.2 Scattering matrix introduction

Chapter 4 was devoted to a discussion of the change of polarization accomplished by several mechanisms. One way to change the polarization of a wave is to scatter the wave by an object. Scattering at a plane interface between two media is an example. The exact way in which the polarization changes (transforms) depends on properties of the object. The aim of radar polarimetry is to utilize the information conveyed in the polarization tranformation properties of an object. The mathematical quantity describing the polarization tranformation properties is called the *scattering matrix* S. This matrix can be defined using the set-up of Fig.5.1. A wave produced by the transmit antenna with polarization $\hat{\mathbf{p}}_t$ illuminates a certain object $A$. This wave induces currents in the object, which give rise to a scattered wave with polarization $\mathbf{p}_s$, generally differing from the polarization of the incident wave. $\mathbf{p}_s$ can be related to $\hat{\mathbf{p}}_t$ by

$$\mathbf{p}_s = \mathbf{S}'\hat{\mathbf{p}}_t = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix} \hat{\mathbf{p}}_t. \tag{5.2}$$

Because the polarization vectors are complex vectors, the elements of the matrix $\mathbf{S}'$ (which is *not* the scattering matrix S) are also complex in general. The polarization vector $\mathbf{p}_s$ of the scattered wave is not normalized, because the scattering process can change the polarization as well as the intensity of the wave.

Let the scattered wave now be incident on a receive antenna with polarization $\hat{\mathbf{p}}_r$. The complex voltage induced in this antenna is given by Eq.(5.1) as

$$V = c\mathbf{p}_s \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{p}}_r = c\mathbf{S}'\hat{\mathbf{p}}_t \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathbf{p}}_r = c\hat{\mathbf{p}}_r \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{S}'\hat{\mathbf{p}}_t \quad (5.3)$$

$$\equiv c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t. \quad (5.4)$$

The complex $2 * 2$ matrix $\mathbf{S}$ is called the scattering matrix. It enables one to compute the complex voltage induced in an arbitrarily polarized receive antenna when the object with scattering matrix $\mathbf{S}$ is illuminated by an arbitrarily polarized transmit antenna. Obviously, the scattering matrix depends on the wavelength of the incident wave, the look-direction with respect to the object etc.

Measurement of the polarization matrix can be done using horizontal and vertical linearly polarized antennas, and registering both amplitude and phase of the received signal voltage. For example, suppose we use a horizontal antenna on transmit, and a vertical one to receive the scattered wave. The received voltage according to Eq.(5.4) is

$$V = c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \mathbf{S}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} = cS_{21} \equiv cS_{vh}.$$

It is therefore natural to rename the scattering matrix element $S_{21}$ to $S_{vh}$. Analog results can be derived for the $hh$, $hv$ and $vv$ cases. So Eq.(5.4) can be rewritten as

$$V = c\hat{\mathbf{p}}_r \cdot \begin{pmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{pmatrix} \hat{\mathbf{p}}_t. \quad (5.5)$$

It is clear from this expression that the scattering matrix can be measured by alternately transmitting a horizontally and a vertically polarized wave, and registering the phase and


Figure 5.1: Set-up used to define the scattering matrix

amplitude of the signal received by a horizontally and vertically polarized antenna. Therefore, when the same antennas can be used for transmitting and receiving (called the monostatic case), two antennas (a horizontal and a vertical one) are sufficient to determine the scattering matrix. This is the way in which the NASA/JPL airborne polarimetric SAR measures the scattering matrix [13].

Let us prove that the squares of the amplitudes of the complex scattering matrix elements are linearly proportional to scattering cross sections. Consider the case of using a horizontally polarized antenna for both transmitting and receiving. According to Eq.(5.5) the voltage $V$ measured at the terminals of the receiving antenna equals $cS_{hh}$. Consequently, the power is proportional to $VV^* \sim S_{hh}S^*_{hh} = |S_{hh}|^2$. But according to the radar formula [14], the power is proportional to the radar cross section, in this case $\sigma_{hh}$. Hence we are led to the conclusion that $\sigma_{hh}$ is proportional to $|S_{hh}|^2$. Similar arguments can be used to prove that $\sigma_{vh}$ is proportional to $|S_{vh}|^2$ etc. Armed with this knowledge the scattering matrix S can be rewritten as

$$\mathbf{S} = \begin{pmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{pmatrix} = c' \begin{pmatrix} \sqrt{\sigma_{hh}}e^{i\phi_{hh}} & \sqrt{\sigma_{hv}}e^{i\phi_{hv}} \\ \sqrt{\sigma_{vh}}e^{i\phi_{vh}} & \sqrt{\sigma_{vv}}e^{i\phi_{vv}} \end{pmatrix}. \tag{5.6}$$

where the phases $\phi$ of the measured voltages and the radar cross sections are made explicit. $c'$ is just another proportionality constant.

It is now clear that the scattering matrix can be seen as a generalized cross section: for the cross section $\sigma$ it was necessary to indicate the polarization of the transmit and receive antennas used, while this information is superfluous for the scattering matrix. In fact, once the scattering matrix is known one is able to compute $\sigma_{xy}$, where $x$ and $y$ are labels for arbitrary antenna polarizations.

## 5.3 General properties of the scattering matrix

In the preceding paragraph it was shown that the complex voltage $V$ as measured by a receive antenna with polarization $\hat{\mathbf{p}}_r$ is given by

$$V = c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t = c'\hat{\mathbf{p}}_r \cdot \begin{pmatrix} \sqrt{\sigma_{hh}}e^{i\phi_{hh}} & \sqrt{\sigma_{hv}}e^{i\phi_{hv}} \\ \sqrt{\sigma_{vh}}e^{i\phi_{vh}} & \sqrt{\sigma_{vv}}e^{i\phi_{vv}} \end{pmatrix} \hat{\mathbf{p}}_t. \tag{5.7}$$

when a wave with with polarization $\hat{\mathbf{p}}_t$ is scattered by an object with scattering matrix S. In principle the complex constant $c$ can be determined such that the amplitude and phase of the voltage $V$ are given correctly by Eq.(5.7). However, several different conventions are used in defining the scattering matrix. In some sources the constant $c$ is incorporated in the scattering matrix. Other sources do not use *normalized* polarization vectors in Eq.(5.7). Still other sources demand, quite arbitrarily, that $S_{hv}$ is real. This can be done harmlessly

because the phase of the voltage $V$ can be simply altered by moving the scattering object along the line of sight. So an overall phase factor (also called *absolute phase*) of the scattering matrix is only related to the distance between te radar and the object, not to any other object feature. The reader just has to bear in mind that the scattering matrices encountered in the literature differ usually by a complex constant only. The constant depends on the scattering matrix definition used. Knowledge of the phase of the constant allows one to compute the distance modulo $\lambda$ to the object. Knowledge of the constant's amplitude allows one to compute the absolute cross section of the object. The constant can be determined by calibration of the system used to measure the scattering matrix. In this report mostly normalized scattering matrices will be used. A normalized scattering matrix is a matrix for which the maximum of $|\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t|$ is 1 (the absolute phase is not determined by this requirement).

We now proceed with a proof demonstrating that the scattering matrix is usually symmetric in the monostatic case (also called the backscatter case). A monostatic radar system is a system for which the positions of the transmit and receive antennas coincide. For the proof we need the fact that the vector inner product $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are column vectors, equals $\mathbf{a}^T\mathbf{b}$ in matrix notation. $^T$ indicates the transpose operation. The column vector $\mathbf{a}$ becomes a row vector by transposing it. So we can rewrite Eq.(5.4) as

$$V = c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t = c\hat{\mathbf{p}}_r^T\mathbf{S}\hat{\mathbf{p}}_t = c\left(\hat{\mathbf{p}}_r^T\mathbf{S}\hat{\mathbf{p}}_t\right)^T = c\hat{\mathbf{p}}_t^T\mathbf{S}^T\hat{\mathbf{p}}_r = c\hat{\mathbf{p}}_t \cdot \mathbf{S}^T\hat{\mathbf{p}}_r.$$

But, according to the reciprocity principle [11], for a reciprocal system (i.e., the radar *and* the medium *and* the scatterer are reciprocal) the voltage $V$ should be the same when the transmit and receive antennas are interchanged. So we may rewrite Eq.(5.4) also as

$$V = c\hat{\mathbf{p}}_t \cdot \mathbf{S}\hat{\mathbf{p}}_r.$$

Because the two preceding derivations are valid for all send and receive polarizations we conclude that $\mathbf{S} = \mathbf{S}^T$ - the scattering matrix is symmetric in the backscatter case. Therefore, $S_{hv} = S_{vh}$. When we exclude the unimportant absolute phase the scattering matrix consists of $8 - 2 - 1 = 5$ independent parameters: three radar cross sections and two phase differences (see Eq.(5.6)).

## 5.4 Scattering matrix examples

The purpose of this paragraph is to find the scattering matrix for a metallic dihedral reflector. Although the introduction of the scattering matrix did not demand that the positions of the send and receive antennas coincided (that is, the introduction was valid for both monostatic and bi-static radars), the discussion is restricted to the monostatic case.

In the remainder of this report it is always assumed that a monostatic radar is used. All remote sensing radars used nowadays are (nearly) monostatic.

Because the derivation of the scattering matrix for a dihedral presents all the features commonly encountered in this kind of derivations a worked-out example will be given for this case only. We will use the *geometric optics approximation*. This approximation treats scattering by assuming that each illuminated point of a scatterer reflects the incident wave as it would be reflected by an infinite plane tangent to the scatterer at that point. So the results of § 4.2 can be used.

Consider Fig.5.2. A dihedral consists of two metal plates connected at right angles to each

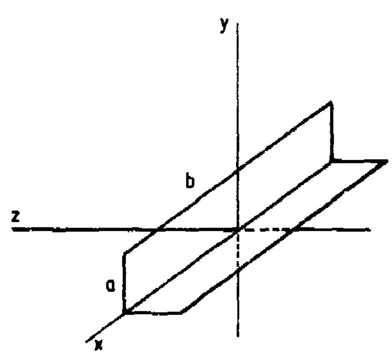


Figure 5.2: Dihedral

other. In the figure the crease is chosen to be aligned with the horizontal $x$-axis, while the two plates are contained in the $xy$- and $xz$-plane. The two identical plates have a height $a$ and a width $b$. It is well-known that the maximum cross-section $\sigma_{hh}$ of such a dihedral is obtained for incidence along the $yz$-plane, while the incidence angle is 45° as measured clockwise from the $y$-axis. The cross section is then given by

$$\sigma_{hh} = \frac{16\pi a^2 b^2}{\lambda^2}. \tag{5.8}$$

for $\lambda$ small compared to $a$ and $b$.

The next thing to do is to investigate the change of polarization accomplished by the dihedral when a wave is incident on it. When the wavelength is assumed to be small compared to the dimensions of the dihedral, we can apply the theory presented in § 4.2 to

this problem. There we found that the Fresnel coefficients were $R_{\parallel} = +1$, $R_{\perp} = -1$ for a metallic medium. This results in Fig.5.3.

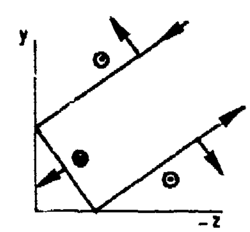


Figure 5.3: Polarization changes caused by scattering at a dihedral

Assume that during the first half period of a wave train the $E_x$ and $E_y$ fields are directed along the positive $x-$ and $y-$axes. The polarizations of the incident wave, the one time reflected wave and the outgoing wave can then be deduced from Fig.5.4, which follows from Fig.5.3.

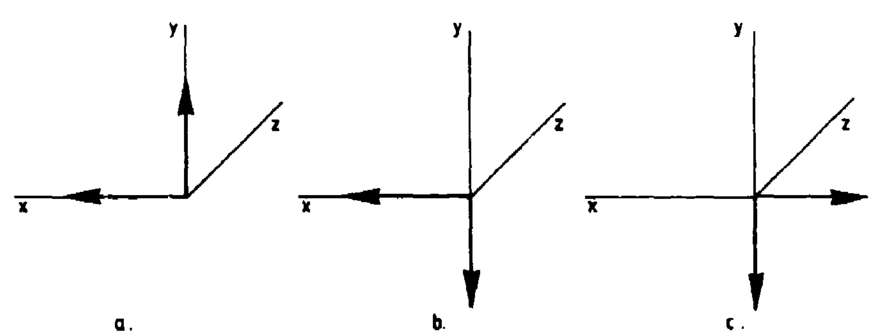


Figure 5.4: Electric field directions during the first half period of the wave train of the incident wave (a), the wave after one reflection (b), and the outgoing wave (c)

Therefore the next two relations should hold for the matrix $\mathbf{S}'$ defined in Eq.(5.2).

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \mathbf{S}' \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \mathbf{S}' \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is interesting to note that the two above equations do not depend on the incidence angle (the angle of the incident wave with respect to the $y$−axis).

From the preceding two equations it can be concluded that the scattering matrix of a dihedral is

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{S}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is a normalized scattering matrix as is easily verified. To remind the reader of the relation between the scattering matrix elements and cross sections, we multiply the former expression by $\sqrt{\sigma_{hh}}$ following from Eq.(5.8), yielding an unnormalized scattering matrix

$$\mathbf{S} = \frac{4\sqrt{\pi}ab}{\lambda} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now it is very clear that $\sigma_{hh} = \sigma_{vv}$ for a dihedral. Furthermore, $\sigma_{hv} = \sigma_{vh} = 0$. Note that the equation is valid for incidence normal to the crease only.

Derivations similar to the one above reveal that the normalized scattering matrices for a flat plate and a trihedral (three flat plates connected at right angles to each other) are identical:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is independent of the aspect angle. This result shows that completely different scatterers can have the same normalized scattering matrix. So radar polarimetry is definitely not the final answer to the inversion problem encountered in remote sensing.

## 5.5 Summary

In the foregoing the scattering matrix was introduced. It was emphasized that each of the scattering matrix elements is linearly proportional to the square root of the corresponding

cross section, for example, $S_{hh}$ is linearly proportional to $\sqrt{\sigma_{hh}}$. It was shown that the scattering matrix can be measured using only two linearly polarized antennas, if one is able to determine the amplitude and phase of the signal induced in the receive antenna by the backscattered wave. It was also noted that several definitions for the scattering matrix exist. Therefore the matrices encountered in the literature can differ by a complex factor. A proof was given for the fact that the scattering matrix is symmetric for the backscatter case when the reciprocity principle is valid. In the last paragraph the scattering matrix for a dihedral was derived, which turned out to be quite simple.

# 6 FURTHER PROPERTIES OF THE SCATTERING MATRIX

This chapter is devoted to some interesting properties of the scattering matrix. First it is shown in § 6.1 that the scattering matrix of an object rotated about the line sight can be determined from the scattering matrix of the un-rotated object and the rotation angle. § 6.2 discusses some properties of radar-symmetric objects. The last two paragaphs solve the problem of minimizing/maximizing the power measured at the terminals of a receiving antenna for backscattering by a given object. This can be used to discriminate or enhance the backscatter of objects with respect to the backscatter of the background.

## 6.1 Rotational dependence of the scattering matrix

Assume that the scattering matrix S of an object is known for a certain aspect angle, frequency etc. When the object is rotated counter-clockwise about the line of sight (the imaginary line connecting the radar with the object) by an angle $\theta$, the scattering matrix will be generally different, say $\mathbf{S}_\theta$. It is maybe rather surprising that $\mathbf{S}_\theta$ can be derived from S and $\theta$. However, a little thought will reveal that it is possible to measure (compute) $\mathbf{S}_\theta$ also by rotating the measurement antennas clockwise, while keeping the object fixed. Keeping this in mind we proceed as follows.

Let a polarization vector p been given by $\begin{pmatrix} a \\ b \end{pmatrix}$. The problem is to determine the polarization vector when this polarization (ellipse) is rotated clockwise by an angle $\theta$. The simplest way to do this is by considering the simple cases of $\mathbf{p}$ being either horizontal polarization $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or vertical polarization $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is easily seen from Fig.6.1 that these polarizations become $\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$ respectively $\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$ after clockwise rotation through an angle $\theta$. Because the polarization vector $\mathbf{p}$ can be rewritten as the sum

$$\mathbf{p} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the rotated vector becomes

$$\mathbf{p}_\theta = a\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} + b\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \equiv \mathbf{R}\mathbf{p}$$

after the clockwise rotation. **R** is an ordinary rotation matrix.

Suppose now that we measure the voltage at the terminals of a receive antenna (polarization $\hat{\mathbf{p}}_r$) when the wave transmitted by the transmit antenna (polarization $\hat{\mathbf{p}}_t$) is backscattered by the counter-clockwise rotated object (scattering matrix $\mathbf{S}_\theta$). This voltage should equal the voltage measured when the object is kept fixed (scattering matrix S), while the antennas are rotated clockwise through an equivalent angle. Therefore, using Eq.(5.4), we conclude that

$$c\hat{\mathbf{p}}_r \cdot \mathbf{S}_\theta \hat{\mathbf{p}}_t = c\hat{\mathbf{p}}_{r,\theta} \cdot \mathbf{S}\hat{\mathbf{p}}_{t,\theta} = c\mathbf{R}\hat{\mathbf{p}}_r \cdot \mathbf{S}\mathbf{R}\hat{\mathbf{p}}_t.$$

But the right-hand side of this equation is equal to

$$c\hat{\mathbf{p}}_r^T \mathbf{R}^T \mathbf{S}\mathbf{R}\hat{\mathbf{p}}_t = c\hat{\mathbf{p}}_r \cdot \left(\mathbf{R}^T\mathbf{S}\mathbf{R}\right)\hat{\mathbf{p}}_t.$$

Because the preceding two equations are valid for all transmit and receive polarizations it follows that the scattering matrix of the rotated object is given by

$$\mathbf{S}_\theta = \mathbf{R}^T\mathbf{S}\mathbf{R} \quad \text{with} \quad \mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{6.1}$$

S denotes the scattering matrix of the un-rotated object, $\theta$ the counter-clockwise rotation angle about the line of sight. For $\theta = 0°$ the rotation matrix **R** becomes the identity matrix. So $\mathbf{S}_{0°} = \mathbf{S}$ as should be the case.

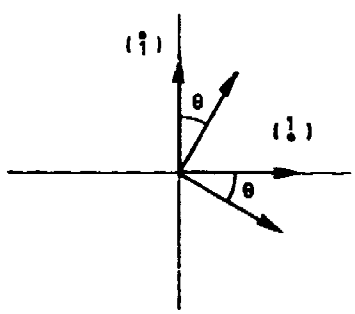


**Figure 6.1: Rotation of polarization vectors**

The foregoing theory was used in [15] to calculate the orientation of a dihedral. The response of the dihedral was measured by an airborne polarimetric synthetic aperture radar flying at a height of about 10 kilometer. The dihedral response was in fact caused by two sides of an erroneously pointing trihedral.

It is interesting to note that the scattering matrix of a trihedral is independent of rotation about the line of sight, because $\mathbf{S}_\phi = \mathbf{R}^T\mathbf{S}\mathbf{R} = \mathbf{R}^T\mathbf{I}\mathbf{R} = \mathbf{R}^T\mathbf{R} = \mathbf{S}$ for this case.

## 6.2 The scattering matrix of objects with one or more symmetry axes

Let an object have a number of radar symmetry axes of which two are non-orthogonal (a radar symmetry axis is a geometrical symmetry axis of the two-dimensional projection of the three-dimensional object onto a plane perpendicular to the line connecting radar and object). Label the angle between these axes $\phi$. Assume that one of the symmetry axes is aligned with the horizontal. When the object is illuminated by a horizontally polarized electrical field a scattered field is produced. This field can be considered to be composed of two contributions, one produced by the upper half, the other produced by the lower half of the object. Because of the symmetry the vertical components of these fields cancel. So the scattered field is horizontally polarized. The scattering matrix of the object is therefore given by the diagonal matrix

$$\mathsf{S} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \tag{6.2}$$

with $a$ and $b$ complex numbers.

When the object is rotated an angle $\theta$ counterclockwise around the line of sight its scattering matrix becomes (using Eq.(6.1))

$$\mathsf{S}(\theta) = \begin{pmatrix} a\cos^2\theta + b\sin^2\theta & \frac{a-b}{2}\sin 2\theta \\ \frac{a-b}{2}\sin 2\theta & a\sin^2\theta + b\cos^2\theta \end{pmatrix}. \tag{6.3}$$

When the angle $\theta$ is chosen such that the second symmetry axis is aligned with the horizontal, the scattering should again be symmetrical. So

$$\frac{a-b}{2}\sin 2\phi = 0 \tag{6.4}$$

should hold. Therefore $a$ should equal $b$, or $\phi$ has to be an integral multiple of $\pi/2$. Because the symmetry axes were assumed to be non-orthogonal, $a$ equals $b$. Substitution in Eq.(6.3) reveals that the normalised scattering matrix is

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.5}$$

When a two-dimensional object has $n$ symmetry axes these have to be rotational symmetry axes, where the angle between two adjacent axes is $180/n$ degrees. This is sketched in Fig.6.2 for $n = 1, 2, 3$ and 4.

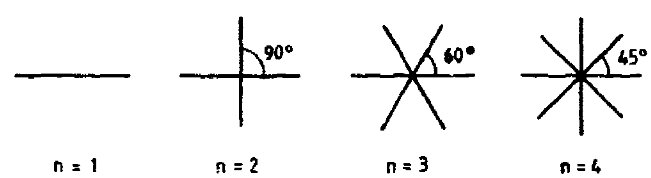


Figure 6.2: Possible configurations for 1, 2, 3 and 4 symmetry axes

It can be concluded that objects with 3 or more symmetry axes have at least one pair of non-orthogonal symmetry axes. So an isosceles triangle, a square and a circle (3, 4 and an inifinite number of symmetry axes respectively) all have a unit scattering matrix.

The simplest examples of objects having a number of symmetry axes of which two or more are non-orthogonal are an isosceles triangle (3 symmetry axes), a square (4 symmetry axes) and a circle (infinite number of symmetry axes).

Most operationally used radar systems utilize identical antenna polarizations at transmit and receive, while offering the possibility to switch the polarization between linear and circular. Uniformly distributed rain can be regarded as a radar target with an infinite number of symmetry axes. The return from the rain should thus vanish theoretically when the radar is operated in the circular mode, because for both left and right circular polarizations

$$V = c\begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = c\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = c\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 0.$$

The return of the object to observe by the radar, for example an airplane, usually does not vanish for this polarization. Therefore the signal to clutter ratio is significantly increased. The fact that the rain echo is, in practice, attenuated by 15 to 30 dB, but not completely suppressed, is usually ascribed to the imperfect sphericity of the raindrops.

Further discussions of symmetry principles and their consequences for the scattering matrix are given in [16,17].

## 6.3 Minimization of the power

The objective of this paragraph is to find the polarizations which minimize the (absolute) voltage appearing in Eq.(5.4) (we will soon see that the minimum is 0 in fact). This is the voltage at the terminals of the receiving antenna. This problem is very easy to solve when the polarizations of the send and receive antenna can be independently controlled. Assume that an object with scattering matrix $\mathbf{S}$ is illuminated by a transmit antenna with polarization $\hat{\mathbf{p}}_t$. Because the receive polarization $\hat{\mathbf{p}}_r$ is independent of the send polarization, it can be chosen to be $(\mathbf{S}\hat{\mathbf{p}}_t)^*_\perp$. But then

$$V = c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t = c(\mathbf{S}\hat{\mathbf{p}}_t)^*_\perp \cdot \mathbf{S}\hat{\mathbf{p}}_t = 0$$

according to the definition of orthogonal polarization Eq.(3.13).

A more challenging problem is to minimize the voltage $V$ subject to the constraint that the send and receive polarizations are equal. It can be seen from Eq.(5.4) that the minimum voltage is 0 for polarizations satisfying

$$\mathbf{S}\hat{\mathbf{p}} = e\hat{\mathbf{p}}^*_\perp \tag{6.6}$$

where $e$ is a complex constant. This is a kind of modified eigenvalue problem, with eigenvalues e. To solve it one needs two intermediate results, which proofs of validity are left to the reader. The first is that an un-normalized polarization vector corresponding to the polarization factor $\rho$ is $\begin{pmatrix} 1 \\ \rho \end{pmatrix}$. The second result is that the polarization factor of the polarization orthogonal to the polarization given by $\rho$ is $-1/\rho^*$. Eq.(6.6) can now be rewritten in terms of the polarization factor $\rho$ of the polarization vector $\hat{\mathbf{p}}$:

$$\begin{pmatrix} S_{hh} & S_{hv} \\ S_{hv} & S_{vv} \end{pmatrix} \begin{pmatrix} 1 \\ \rho \end{pmatrix} = e \begin{pmatrix} 1 \\ \frac{-1}{\rho} \end{pmatrix}.$$

The scattering matrix is assumed to be symmetric, which is generally so for a reciprocal system. This equation is readily transformed into the cubic equation

$$S_{vv}\rho^2 + 2S_{hv}\rho + S_{hh} = 0 \tag{6.7}$$

with solutions

$$\rho_{1,2} = \frac{-S_{hv} \pm \sqrt{S_{hv}^2 - S_{hh}S_{vv}}}{S_{vv}}. \tag{6.8}$$

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$$\begin{pmatrix} S_{hh} & S_{hv} \\ S_{hv} & S_{vv} \end{pmatrix} \begin{pmatrix} 1 \\ \rho \end{pmatrix} = e \begin{pmatrix} 1 \\ \frac{-1}{\rho} \end{pmatrix}.$$

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with solutions

$$\rho_{1,2} = \frac{-S_{hv} \pm \sqrt{S_{hv}^2 - S_{hh}S_{vv}}}{S_{vv}}. \tag{6.8}$$

This demonstrates that when the same antenna is used on transmit and receive, it is nevertheless possible to make the return from a object to vanish. There are in general two polarizations that will accomplish this. The polarization factors of these two polarizations are given by Eq.(6.8). These polarizations are called the co-polarized nulls [18].

## 6.4 Maximization of the power

As a logical continuation of the foregoing this paragraph is devoted to the maximization problem. The problem is to find the polarizations $\hat{\mathbf{p}}_t$, $\hat{\mathbf{p}}_r$ which maximize

$$VV^* = |c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_t|^2 \tag{6.9}$$

It will be proven that the maximum power received can be attained with only one antenna used for transmitting and receiving, just like in the minimization case of the preceding paragraph. This polarization is again found as a solution to a kind of eigenvalue problem. The proof uses the theory developed in [18], which solved the problem of maximizing the power in the backscattered wave. However, here it is also proven that the receive polarization matching the polarization of the maximized backscattered wave equals the polarization that maximizes the return in the backscattered wave.

First *assume* that $\hat{\mathbf{p}} = \hat{\mathbf{p}}_t = \hat{\mathbf{p}}_r$ maximizes $VV^*$. It is then clear from Eq.(6.9) that $\hat{\mathbf{p}}$ should satisfy

$$\mathbf{S}\hat{\mathbf{p}} = \epsilon\hat{\mathbf{p}}^*, \tag{6.10}$$

again a kind of eigenvalue problem (but not in the ordinary form, because of the complex conjugation in the right-hand side of the equation). Nevertheless, $\epsilon$ will be called the eigenvalue, and $\hat{\mathbf{p}}$ the eigenvector. In general, there are two solutions to this equation:

$$\begin{aligned} \mathbf{S}\hat{\mathbf{p}}_1 &= \epsilon_1\hat{\mathbf{p}}_1^* \\ \mathbf{S}\hat{\mathbf{p}}_2 &= \epsilon_2\hat{\mathbf{p}}_2^*. \end{aligned}$$

Since the matrix $S$ is symmetric, we have

$$|\mathbf{S}\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2| = |\hat{\mathbf{p}}_1 \cdot \mathbf{S}\hat{\mathbf{p}}_2|.$$

On combining the last three equations we get

$$|e_1||\hat{\mathbf{p}}_1^* \cdot \hat{\mathbf{p}}_2| = |e_2||\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^*|.$$

Hence, if $|e_1| \neq |e_2|$,

$$\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^* = 0.$$

But, according to Eq.(3.13), these two eigenvectors are orthogonal. The two eigenvectors thus form an orthonormal set, because they are orthogonal and both normalized. When the two eigenvector polarizations are displayed on the Poincaré sphere they are antipodal.

It is now possible to show that the maximum *return* is obtained when the polarization of the eigenvector with the absolute largest eigenvalue is used. By the return the power in the backscattered wave is meant. To prove this we write an arbitrary polarization vector $\hat{\mathbf{q}}$ as a linear combination of the two eigenvectors:

$$\hat{\mathbf{q}} = a_1\hat{\mathbf{p}}_1 + a_2\hat{\mathbf{p}}_2 \tag{6.11}$$

When the object with scattering matrix $\mathbf{S}$ is illuminated by a transmit antenna with polarization $\hat{\mathbf{q}}$, the power $P_{wave}$ in the backscattered wave is proportional to

$$\mathbf{S}'\hat{\mathbf{q}} \cdot \mathbf{S}'^*\hat{\mathbf{q}}^* = \mathbf{S}\hat{\mathbf{q}} \cdot \mathbf{S}^*\hat{\mathbf{q}}^*.$$

Note the use of $\mathbf{S}'$, which is necessary because of the defininition of the scattering matrix (Eq.(5.4)). On using expression (6.11) one obtains

$$\begin{aligned} P_{wave} &\propto (a_1\mathbf{S}\hat{\mathbf{p}}_1 + a_2\mathbf{S}\hat{\mathbf{p}}_2) \cdot (a_1^*\mathbf{S}^*\hat{\mathbf{p}}_1^* + a_2^*\mathbf{S}^*\hat{\mathbf{p}}_2^*) = |a_1|^2|e_1|^2 + |a_2|^2|e_2|^2 && (6.12)\\ &= |e_1|^2 - |a_2|^2\left(|e_1|^2 - |e_2|^2\right). && (6.13)\end{aligned}$$

Without loss of generality one may assume that $|e_1| > |e_2|$. But then it becomes clear that the maximum return $P_{wave}$ is attained when $a_2 = 0$. The maximum value is $|e_1|^2$, while the polarization for which this maximum is attained is given by the eigenvector $\hat{\mathbf{p}}_1$. Besides this, it can also been concluded from Eq.(6.13) that the minimum return is obtained when $|a_2| = 1$. The minimum return $|e_2|^2$ is obtained when the object is illuminated by a wave with the polarization of the eigenvector $\hat{\mathbf{p}}_2$.

This is not yet the end of the proof, because we now have only shown that the power in the backscattered wave is maximal, when the object is illuminated by a wave with polarization $\hat{\mathbf{p}}_1$. Now the power at the terminals of the receive antenna with polarization given by $\hat{\mathbf{p}}_r$ has to be maximized, that is,

$$P = |c\hat{\mathbf{p}}_r \cdot \mathbf{S}\hat{\mathbf{p}}_1|^2$$

should be maximized.

When this condition is met, the polarization of the receive antenna is matched to the polarization of the backscattered wave. But, because $\mathbf{S}\hat{\mathbf{p}}_1 = e_1\hat{\mathbf{p}}_1^*$, it is clear that $\hat{\mathbf{p}}_r$ must equal $\hat{\mathbf{p}}_1$. This completes the proof. The polarization $\hat{\mathbf{p}}_1$ is called the co-polarized maximum polarization.

Now Eq.(6.10) is solved for $\hat{\mathbf{p}}$. Analogously to the approach used in the preceding paragraph we rewrite the polarization vector as $\begin{pmatrix} 1 \\ \rho \end{pmatrix}$. Substitution of this vector in Eq.(6.10) gives

$$S_{hv}\left(1 - |\rho|^2\right) + \rho S_{vv} - \rho^* S_{hh} = 0.$$

This equation is not as easy to solve as the cubic equation (6.7). However, it turns out that after some algebraic manipulations the equation can be rewritten as the cubic equation

$$a\rho^2 - b\rho - a^* = 0, \tag{6.14}$$

where

$$a = S_{hh}^* S_{hv} + S_{vv} S_{hv}^* \text{ and } b = |S_{vv}|^2 - |S_{hh}|^2.$$

The solutions of this equation in terms of $a$ and $b$ are

$$\rho_{1,2} = \frac{b \pm \sqrt{b^2 + 4|a|^2}}{2a}. \tag{6.15}$$

Because $\rho_1\rho_2^* = -1$ the polarizations corresponding to these two solutions are orthogonal, as predicted.

Transmitting and receiving with the polarization factor corresponding to the largest eigenvalue of the two solutions gives maximum output at the terminals of the receiving antenna.

Fig.6.3 shows the four solutions of Eq.(6.8) and Eq.(6.15) as displayed on the Poincaré sphere. Huynen [18] proved that the four solutions form a fork contained in a plane when displayed on the sphere. As predicted, the solutions of Eq.(6.15) are situated antipodal on the sphere. The two prongs of the fork correspond to the co-polarized nulls. The handle of the fork corresponds to the co-polarized maximum polarization. The angle between the two co-polarized nulls is bisected by the line connecting the solutions of Eq.(6.15).

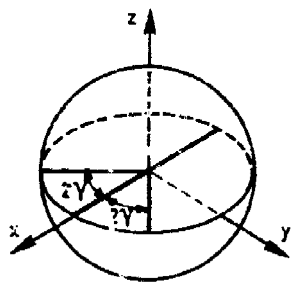


Figure 6.3: Huynen's fork (from [5])

## 6.5 Summary

When an object is rotated about the line of sight its scattering matrix changes in general. The rotational dependence of the scattering matrix was explored in the first paragraph. It turned out that when the scattering matrix of an object is known, the scattering matrix of the same object rotated through a certain angle about the line of sight is given by Eq.(6.1).

The scattering matrix of a radar-symmetric object is diagonalized by rotating it about the line of sight until its symmetry plane is either horizontal or vertical. Objects with more than two radar symmetry axes have a unit scattering matrix.

In the last two paragraphs the problem of minimizing and maximizing the power at the output terminals of a receiving antenna was solved. It was shown that for the minimization problem two solutions exist where the receive and transmit antenna polarizations are equal. The polarizations are called the co-polarized nulls. The minimum power is zero. For the maximization problem one solution exists with equal send and receive polarizations. This polarization is called the co-polarized maximum. When these three 'characteristic' polarizations are mapped onto the Poincaré sphere, they form a fork.

# 7 THE STOKES VECTOR

In the foregoing the discussion was restricted to *completely* polarized waves. However, the waves commonly encountered in nature are most often (if not always) *partially polarized*. Partially polarized waves are waves with a time-varying polarization. They are (partially !) described by the Stokes vector, instead of the polarization vector. This chapter introduces the Stokes vector and a related quantity, the degree of polarization.

## 7.1 Partially polarized waves

By definition, a completely polarized wave is a wave for which the polarization vector

$$\mathbf{p} = \begin{pmatrix} E_x \\ E_y e^{i\delta} \end{pmatrix} \text{ with } \delta \equiv \delta_y - \delta_x \tag{7.1}$$

is independent of time. So $E_x, E_y$ and $\delta_y - \delta_x$ are constants. The polarization ellipse of a completely polarized wave has constant orientation, ellipticity and size. This implies that a completely polarized wave is represented by a single point on the Poincaré sphere. A radar transmits an almost completely polarized wave.

The other extreme is constituted by the *completely unpolarized* waves. Suppose one measures the polarization of the light emitted by an ordinary gas-discharge tube. The light is produced by a great number of decaying atoms. Some time later ($\gg$ the mean decay time of the atoms) the light is produced by a completely different set of decaying atomes. Consequently, the polarization of this light is not related to the polarization measured earlier. The orientation and ellipticity of the polarization ellipse vary wildly in time. In fact they are randomly distributed within their respective ranges. The intensity fluctuates also. When the polarizations resulting from a lot of measurements are displayed on the Poincaré sphere, a shell results.

Between these two extremes one finds the *partially* polarized waves. For partially polarized waves the polarization ellipse also varies in time, but not completely at random. A more or less sharply defined mean ellipse can be recognized. Displaying the varying polarization of a partially polarized wave on the Poincaré sphere gives a cluster of points on the sphere. Because the intensity generally fluctuates also, the distance of the points to the centre of the sphere is not constant. So in Poincaré space a partially polarized wave is characterized by a cloud of points.

A partially polarized wave is generally obtained when a completely polarized wave is scattered by a target which varies in time. An example is given in Fig.7.1.

These figures were obtained by illuminating three different objects (rain clutter, ground clutter and an aircraft) with a horizontally (completely) polarized 5 GHz wave. The intensity and polarization of the backscattered wave were measured at several instances in time. This resulted for the intensity in the three histograms. The polarizations are displayed on a certain projection of the Poincaré sphere. Some polarizations are indicated. It can be seen that the intensity histograms differ for the three different objects. The polarizations of the backscattered wave cluster on the Poincaré sphere. The positions of the three clusters differ. The spread differs also: it is biggest for the rain clutter and smallest for the ground clutter. At first sight it seems strange that the polarization of the wave backscattered by a fixed object like the aircraft fluctuates in time. However, small changes in the aspect angle can result in large variations. So small changes in the orientation of the aircraft, possibly caused by the wind, can account for the spread.

It is also possible to measure the polarization and intensity of a wave backscattered by different parts of a homogeneous area, like an agricultural field. When the results are displayed as in Fig.7.1 similar results are obtained.

The mean position of a cluster, its dimensions or other statistical quantities can be used to discriminate between objects, as is clear from Fig.7.1. The Stokes vector, which measures the mean position and the overall dimension of the 'cloud' for partially polarized waves, is mostly used.

## 7.2 The Stokes vector for completely polarized waves

The Stokes vector of a completely polarized wave is defined by

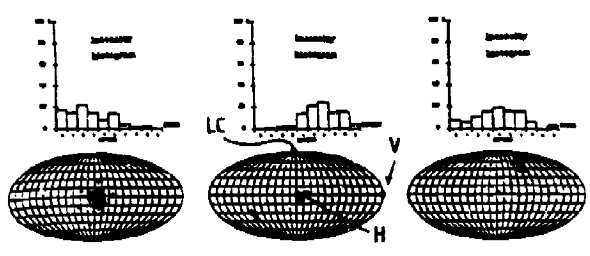


Figure 7.1: Three partially polarized waves (from [6])

$$\mathbf{s} \equiv \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv \begin{pmatrix} E_x^2 + E_y^2 \\ E_x^2 - E_y^2 \\ 2E_xE_y\cos\delta \\ 2E_xE_y\sin\delta \end{pmatrix}, \tag{7.2}$$

where the parameters $E_x$, $E_y$ and $\delta$ are those of Eq.(7.1). $s_0$ is proportional to the intensity of the wave (see Eq.(2.3)). It is readily checked that $s_0^2 = s_1^2 + s_2^2 + s_3^2$. Consequently, the four-dimenional Stokes vector of a completely polarized wave consists of only three independent parameters.

The Stokes vector can also be expressed in terms of the geometrical parameters $\psi$ and $\chi$, and $E_x^2 + E_y^2$ (3 independent parameters !). By using Eqs.(3.2) ... (3.5) it is possible to show that

$$\mathbf{s} = s_0 \begin{pmatrix} 1 \\ \cos(2\chi)\cos(2\psi) \\ \cos(2\chi)\sin(2\psi) \\ \sin(2\chi) \end{pmatrix} \text{ with } s_0 = E_x^2 + E_y^2. \tag{7.3}$$

This equation reveals the correspondence between the Stokes vector and the Poincaré sphere presentations. The components of the Stokes vector are the coordinates in a rectangular coordinate system, as shown in Fig.3.3. The Stokes vector is often used, because its components are easily measured. In fact, it can be measured by doing power measurements only. When we let for example $P$(horizontal) denote the power resulting from a measurement with a horizontally polarized antenna, we can write

$$\mathbf{s} \propto \begin{pmatrix} P(\text{horizontal}) & + & P(\text{vertical}) \\ P(\text{horizontal}) & - & P(\text{vertical}) \\ P(\text{linear } 45^\circ) & - & P(\text{linear } 135^\circ) \\ P(\text{left circular}) & - & P(\text{right circular}) \end{pmatrix}.$$

To prove this, Eq.(5.1) must be used. An example is given here for $s_3$. $P$(left circular) is computed as follows:

$$\begin{aligned} P(\text{left circular}) \;\; &\propto \;\; |V(\text{left circular})|^2 = \left| c \begin{pmatrix} E_x \\ E_y e^{i\delta} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \right|^2 \\ &= \left| c\left(E_x - iE_y e^{i\delta}\right)\right|^2 = |c|^2 \left(E_x^2 + E_y^2 + 2E_xE_y\sin\delta\right) \end{aligned}$$

Following the same procedure for $P$(right circular) one gets

$$P(\text{right circular}) = |c|^2 \left(E_x^2 + E_y^2 - 2E_xE_y \sin\delta\right).$$

Therefore,

$$s_3 \propto P(\text{left circular}) - P(\text{right circular}) \propto 2E_xE_y \sin\delta,$$

in accordance with the definition of the Stokes vector of Eq.(7.2).

When the Stokes vector representation is included, we have now four quantities giving the polarization of a completely polarized wave:

- the orientation $\psi$ and ellipticity $\chi$.
- the normalized polarization vector $\hat{\mathbf{p}}$. When the polarization vector is not normalized, like $\mathbf{p}$, the intensity of the wave can be computed.
- the polarization factor $\rho$.
- the Stokes vector $\mathbf{s}$. Component $s_0$ is proportional to the intensity of the wave. A Stokes vector is 'normalized' by dividing the four components by $s_0$. It then consists of only two independent parameters, giving the polarization.

## 7.3 The Stokes vector for partially polarized waves

The Stokes vector for a partially polarized wave is simply defined as the time average of the Stokes vector for the completely polarized wave of Eq.(7.2):

$$\mathbf{s} \equiv \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv \begin{pmatrix} \langle E_x^2\rangle + \langle E_y^2\rangle \\ \langle E_x^2\rangle - \langle E_y^2\rangle \\ 2\langle E_xE_y\cos\delta\rangle \\ 2\langle E_xE_y\sin\delta\rangle \end{pmatrix}, \text{ where } \langle\cdots\rangle \equiv \frac{1}{2T}\lim_{T\to\infty}\int_{-T}^{T}\langle\cdots\rangle\, dt \qquad (7.4)$$

Let us prove that $s_0^2 \geq s_1^2 + s_2^2 + s_3^2$, where the equality holds for completely polarized waves only. To do this define

$$\Delta = s_0^2 - s_1^2 - s_2^2 - s_3^2 = 4\left\{\langle E_x^2\rangle\langle E_y^2\rangle - \langle E_xE_y\cos\delta\rangle^2 - \langle E_xE_y\sin\delta\rangle^2\right\}.$$

Because $|\sin\delta|$ and $|\cos\delta|$ are always less than or equal to one, and greater than or equal to zero

$$\langle E_x E_y \cos\delta\rangle^2 + \langle E_x E_y \sin\delta\rangle^2 \le \langle E_x E_y\rangle^2 \le \langle E_x^2\rangle\langle E_y^2\rangle$$

holds, where the Schwarz inequality [19] was used for the second step. So $\Delta$ is always positive or zero. When $E_x$, $E_y$ and $\delta$ are constants $\Delta = 0$. This completes the proof.

Because the components of the Stokes vector are (averaged) powers, the Stokes vector of a sum of independent waves is the sum of the Stokes vectors of the individual waves. By independent is meant that there are no permanent phase relations between the individual waves.

## 7.4 Degree of polarization

A completely unpolarized wave has a Stokes vector

$$\mathbf{s} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $a$ is a real, positive constant. A proof for this can be found in [3]. Loosely speaking, one can state that $s_1$, $s_2$ and $s_3$ are zero, because $E_x$, $E_y$ and $\delta$ fluctuate so wildly for a completely unpolarized wave that the three components average to zero.

Any partially polarized wave can be regarded as the sum of a completely unpolarized and a completely polarized wave, i.e.

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = s_0 \begin{pmatrix} 1-d \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} s_0 d \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}. \tag{7.5}$$

The first term of the sum represents a completely unpolarized wave, the second part a completely polarized wave. Therefore $d$ must be chosen so that $(s_0 d)^2 = s_1^2 + s_2^2 + s_3^2$, a necessary condition for complete polarization. But then

$$d = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0}. \tag{7.6}$$

$d$ is the ratio of the intensity of the completely polarized part and the intensity of the total wave.

The Stokes vector of a partially polarized wave has four independent components, while the Stokes vector for a completely polarized wave has only three. Three of the four parameters of a partially polarized wave are the average position of the cloud of points mentioned in §1 of this chapter. The fourth parameter is often taken to be the *degree of polarization*. It is related to the overall dimension of the cloud. The degree of polarization is defined as the parameter $d$ of Eq.(7.6). From this equation it follows that $d$ is 0 for a completely unpolarized wave, 1 for a completely polarized wave, and in between these values for a partially polarized wave.

## 7.5 Summary

When the varying polarization of a partially polarized wave is displayed in Poincaré space, a more or less sharply defined cloud of points results. The four-dimensional Stokes vector can be used to describe some (but not all!) of the features of this cloud. Each of the components of the vector can be determined by doing power measurements. In fact, the components *are* powers. Therefore, additivity holds: the Stokes vector of a sum of independent waves equals the sum of the Stokes vectors of the individual waves. The degree of polarization, which can be determined from the Stokes vector, is a measure for the variability of the polarization in time (or in space, when the Stokes vector is a space average).

# 8 THE STOKES MATRIX

The scattering matrix relates the polarization of the wave scattered by a stationary object to the polarization of the incident wave. It is also possible to compute the voltage induced in a receive antenna by the backscattered wave. The wave scattered by a time-varying object is partially polarized, which can be characterized by a Stokes vector. It is therefore natural to introduce the Stokes matrix which can be used to formulate similar relations, but now for time-varying objects in terms of Stokes vectors. After development of the Stokes matrix formulation the polarization signature is introduced. The polarization signature is a plot of the information contained in the Stokes matrix. Finally, attention is paid to minimization/maximization problems similar to those of §6.3 and §6.4.

## 8.1 The Stokes matrix of a stationary object

To define the Stokes matrix we first have to recall Eq.(5.5):

$$V = c\hat{\mathbf{p}}_r \cdot \begin{pmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{pmatrix} \hat{\mathbf{p}}_t. \tag{8.1}$$

This equation gives the voltage at the terminals of a receive antenna with polarization $\hat{\mathbf{p}}_r$, when an object with scattering matrix $\mathbf{S}$ is illuminated by a transmit antenna with polarization $\hat{\mathbf{p}}_t$. The scattering matrix is symmetric in the backscatter case, when the positions of the antennas coincide.

The Stokes matrix $\mathbf{M}$ is defined completely analogously to Eq.(8.1) by

$$P = c\mathbf{s}_r \cdot \mathbf{M}\mathbf{s}_t. \tag{8.2}$$

This equation gives the power $P$ ($\propto VV^*$) delivered into a load connected to the terminals of a receive antenna with polarization given by the Stokes vector $\mathbf{s}_r$, when an object with Stokes matrix $\mathbf{M}$ is illuminated by a transmit antenna with polarization $\mathbf{s}_t$. $c$ is again a proportionality constant. The Stokes vectors belong to completely polarized waves, because they represent antenna polarizations. Because a Stokes vector has four real components, the $4 * 4$ square matrix $\mathbf{M}$ is real, as is $c$.

The Stokes matrix for a stationary scatterer can be derived from the scattering matrix of that scatterer. It turns out that the elements of the symmetric Stokes matrix are related to the scattering matrix elements by

$$
\begin{aligned}
M_{11} &= \frac{1}{4}\left(|S_{hh}|^2 + |S_{vv}|^2 + 2|S_{hv}|^2\right) && (8.3)\\
M_{12} &= \frac{1}{4}\left(|S_{hh}|^2 - |S_{vv}|^2\right) && (8.4)\\
M_{13} &= \frac{1}{2}\mathrm{Re}(S_{hh}^{*}S_{hv} + S_{vv}^{*}S_{hv}) && (8.5)\\
M_{14} &= \frac{1}{2}\mathrm{Im}(S_{hh}^{*}S_{hv} - S_{vv}^{*}S_{hv}) && (8.6)\\
M_{22} &= \frac{1}{4}\left(|S_{hh}|^2 + |S_{vv}|^2 - 2|S_{hv}|^2\right) && (8.7)\\
M_{23} &= \frac{1}{2}\mathrm{Re}(S_{hh}^{*}S_{hv} - S_{vv}^{*}S_{hv}) && (8.8)\\
M_{24} &= \frac{1}{2}\mathrm{Im}(S_{hh}^{*}S_{hv} + S_{vv}^{*}S_{hv}) && (8.9)\\
M_{33} &= \frac{1}{2}|S_{hv}|^2 + \frac{1}{2}\mathrm{Re}(S_{hh}^{*}S_{vv}) && (8.10)\\
M_{34} &= \frac{1}{2}\mathrm{Im}(S_{hh}^{*}S_{vv}) && (8.11)\\
M_{44} &= \frac{1}{2}|S_{hv}|^2 - \frac{1}{2}\mathrm{Re}(S_{hh}^{*}S_{vv}) && (8.12)
\end{aligned}
$$

For example, using these equations, the Stokes matrix of a flat plate (unit scattering matrix) is readily computed to be

$$
\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (8.13)
$$

The symmetric scattering matrix contains 5 independent parameters when the absolute phase is neglected (§5.3). The Stokes matrix, which can be measured by power measurements only, should thus contain also 5 independent parameters (the absolute phase is not determined by a power measurement). The Stokes matrix is symmetric and therefore contains at most $16 - 6 = 10$ independent parameters. It can be seen from Eqs.(8.3) ... (8.12) that the following 5 independent relations exist between the elements of the Stokes matrix:

$$
\begin{aligned}
M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2 &= M_{11}^2 - M_{22}^2 && (8.14)\\
M_{13}^2 - M_{14}^2 - M_{23}^2 + M_{24}^2 &= M_{33}^2 - M_{44}^2 && (8.15)\\
M_{13}M_{23} + M_{14}M_{24} &= M_{12}(M_{11} - M_{22}) && (8.16)\\
M_{13}M_{14} - M_{23}M_{24} &= M_{34}(M_{33} + M_{44}) && (8.17)\\
M_{11} &= M_{22} + M_{33} + M_{44} && (8.18)
\end{aligned}
$$

This makes the number of independent elements $10 - 5 = 5$, as expected.

## 8.2 The Stokes matrix of a time-varying object

The wave scattered by a time-varying object is partially polarized. The polarization of the wave thus fluctuates in time. As a consequence, its scattering matrix and corresponding Stokes matrix are also time-dependent. The proper way to describe a time-varying object is to measure its Stokes matrix repeatedly, and average the results. This is valid, because the elements of the matrix result from (possibly indirect) power measurements. Of the five relations given by Eqs.(8.14) ... (8.18) only the last one is linear. When a number of Stokes matrices corresponding to stationary objects is summed, only this relation will remain to be valid for the sum. The number of independent elements of a Stokes matrix for a time-varying object is therefore $5 + 4 = 9$. The class of time-varying objects is thus larger than the class of stationary objects. It is clear that there corresponds no scattering matrix to the averaged Stokes matrix of a time-varying object.

The image resulting from a measurement by a polarimetric imaging radar consists of pixels for which the scattering matrix is measured. The Stokes matrix of a distributed object, like an agricultural field, is determined in two steps. First the scattering matrices of the object pixels are converted to Stokes matrices using Eq.(8.3) ... (8.12). These Stokes matrices are then summed to give the Stokes matrix characterizing the distributed object. Note that the time average is replaced here by a space average.

## 8.3 Polarization signatures

A problem is to visualize polarimetric data, i.e., Stokes matrices. As can be seen from Eq.(8.2) the power received depends on the Stokes matrix and the send- and receive polarizations. In principle five dimensions (4 for the send and receive polarization and 1 for the power) are needed to visualize the contents of a given Stokes matrix. Several simplifying approaches have been used to display part of the information [18]. In [5] so-called co-polarization signatures were introduced, defined as follows: assume the send- and receive polarizations of a radar to be identical. The co-polarization signature is a three-dimensional plot of the received power of Eq.(8.2) as a function of the orientation and ellipticity of the send/receive polarization. For example, the co-polarization signature of a flat plate at normal incidence is given by

$$P \propto \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = 1 + s_1^2 + s_2^2 - s_3^2$$

$$= \quad 1+\cos^2(2\chi)\cos^2(2\psi)+\cos^2(2\chi)\sin^2(2\psi)-\sin^2(2\chi)=1+\cos(4\chi),$$

where Eqs.(7.4), (8.2) and (8.13) were used. The signature is shown in Fig.8.1.

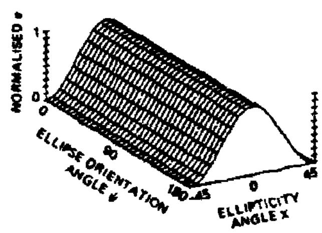


Figure 8.1: Co-polarization signature of a flat plate at normal incidence (from [7])

The height of a point on the surface of a signature is proportional to received power, and consequently to the radar cross section also. It can be seen that all linear polarizations, characterized by $\chi = 0$, give maximum power. The power is zero for all circular polarizations ($\chi = \pm 45°$).

The co-polarization signature has been extensively used to analyze the data obtained for San Francisco by a polarimetric SAR [7]. Fig.8.2 presents an example. It shows the signature of an urban area together with the signature of a dihedral. The signature of the urban area looks like that of the dihedral with an offset added. This suggests that the scattering from the urban area is caused by a two-bounce scattering mechanism.

In addition to the co-polarized signature one can define the *cross-polarized* signature. The only difference is the fact that the two antennas are cross-polarized in this case.

## 8.4 Minimization and maximization of the received power

§6.3 and §6.4 handled the minimization and maximization problem for the power scattered by a stationary object, which can be characterized by a scattering matrix. A time-varying or distributed object is characterized by an average Stokes matrix for which no corresponding scattering matrix exists. The minimization/maximization problem is therefore different. In contrast to the stationary case, no co-polarized solutions exist in this case: the extremes are generally attained for non-equal send and receive polarizations.

Nevertheless, in [20] the problem is solved for the case where the transmit and receive polarizations are identical. So the extremes of $P(\mathbf{s}) = \mathbf{s}\cdot\mathbf{M}\mathbf{s}$ are determined for a given

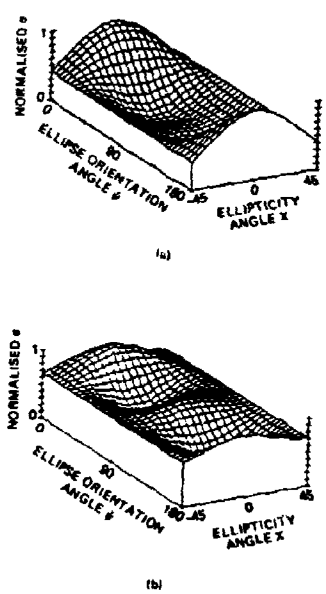


Figure 8.2: Co-polarization signatures of an urban area (a) and a dihedral (b) (from [7])

Stokes matrix **M**. This corresponds to locating the extremes (including saddle-points and local extremes) of the co-polarized signature. The method used leads to a sixth-order polynomial. Each of the real roots of the polynomial corresponds to an extreme. It is proven that there are at least two roots, so there are $2, 3 \ldots 6$ extremes. Generally, the minimum power is not zero because the backscattered wave is partially polarized. Minimum power is received from a partially polarized wave when the polarization of the receiving antenna is orhogonal to the polarization of the completely polarized part of the wave (see §7.4). The minimum power is half of the power contained in the completely unpolarized part of the wave [21]. It shows up in the co-polarization signature as a 'pedestal'. All points of the signature are located above or on this pedestal.

The problem of minimizing/maximizing the power in Eq.(8.2) as a function of the independent send and receive polarizations is attacked in [5]. The solution leads to a system of two non-linear equations which is not easy to solve.

It is often necessary to optimize the contrast between two different objects with Stokes matrices $\mathbf{M}_1$ and $\mathbf{M}_2$. That is, the polarizations $\mathbf{s}_t$ and $\mathbf{s}_r$ have to be determined which minimize or maximize the ratio

$$\frac{P_1}{P_2} = \frac{\mathbf{s}_r \cdot \mathbf{M}_1 \mathbf{s}_t}{\mathbf{s}_r \cdot \mathbf{M}_2 \mathbf{s}_t}.$$

A hybrid solution is given in [22]. By hybrid is meant that the method proposed is half analytical and half numerical. The analytical part consists of an expression giving the optimal receive polarization $\mathbf{s}_r$ for a fixed send polarization $\mathbf{s}_t$. The numerical part consists of computing this $\mathbf{s}_r$ for several values of $\mathbf{s}_t$ until the ratio of $P_1$ and $P_2$ attains its minimum or maximum. In practice the transmit polarization is varied over a sufficiently dense grid of orientation and ellipticity angles.

## 8.5 Summary

In this chapter the Stokes matrix was introduced. While the scattering matrix formulation uses electric fields (polarization vectors), the Stokes matrix formulation uses Stokes vectors. The polarization properties of a stationary object are represented by the scattering matrix. A time-varying or distributed object is represented by an average Stokes matrix. A co-polarization signature can be used to display (part of) the information contained in the Stokes matrix. This signature is a three-dimensional plot of the power received by a radar with identical receive and tranmit antennas, for an object with a given Stokes matrix. An outline of some solutions to minimization and maximization problems was given in the last paragraph.

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FOR REMOTE SENSING OF THE EARTH'S SURFACE FROM AIRBORNE OR SPACEBORNE PLATFORMS ONE CAN USE RADAR SYSTEMS. BECAUSE THE RADAR ITSELF ILLUMINATES THE EARTH SURFACE ONE CAN PERFORM POLARIMETRIC MEASUREMENTS IN PRINCIPLE. THIS IMPLIES THAT ONE MEASURES HOW THE POLARIZATION OF THE SCATTERD WAVE DEPENDS ON THE POLARIZATION OF THE ILLUMINATING WAVE. THIS DEPENDENCY GENERALLY DIFFERS FOR DIFFERENT SCATTERING SURFACES, AND THUS CONTAINS INFORMATION ABOUT THOSES SURFACES. THE AIM OF RADAR POLARIMETRY IS TO UTILIZE THE INFORMATION ONE CAN GET BY PERFORMING POLARIMETRIC MEASUREMENTS WITH RADAR.
THIS REPORT CONTAINS A GENTLE INTRODUCTION INTO RADAR POLARIMETRY. IT PRESENTS THE NECESSARY DEFINITIONS TOGETHER WITH EXAMPLES OF THEIR IMPLICATIONS AND PRACTICAL USE. AFTER STUDYING THIS REPORT ONE SHOULD BE ABLE TO UNDERSTAND THE RECENT LITERATURE ON THE SUBJECT.

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